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Periodic Solution for Nonlinear System of Differential Equations of some Operators with Impulsive Action

Raad N. Butris

Department of Mathematics, Collage of Basic Education, University of Duhok, Kurdistan Region, Iraq
Email:raad.butris@uod.ac

ABSTRACT

In this paper we investigate the periodic solution for nonlinear system of differential equations of some operators with impulsive action, by using the numerical-analytic method for periodic solutions which is given by Samoilenko. Theorems on existence and uniqueness of solutions are established under some necessary and sufficient conductions on compact space. This investigation leads us to the improving and extending to the above method and expands the results gained by Butris.

Key words: Numerical-analytic method, periodic solutions, differential equations, impulsive action of operators

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1. INTRODUCTION

There are many subjects in physics and technology using mathematical methods that depends on the nonlinear differential equations, and it became clear that the existence of the periodic solutions and it's algorithm structure from more important problems in the present time. Where many of studies and researches dedicates for treatment the autonomous and non-autonomous periodic systems and specially with the integral equations and differential equations and the linear and nonlinear differential and which is dealing in general shape with the problems about periodic solutions theory and the modern methods in its quality treatment for the periodic differential equations. Samoilenko [6] andothersthey usednumerical analytic method to study the periodic solutions of differential equation and this method include uniformly sequences of periodic functions in studies [2-10].

Butris [2] assumes the numerical analytic method to study the periodic solutions for the ordinary differential equations and its algorithm structure and this method include uniformly sequences of the periodic functions and the result of that study is the using of the periodic solutions on a wide range for example see the following system of nonlinear differential equation, which has the form:

$\frac{dx}{dt} = \lambda x + f(t,x,y) \quad , \quad t \neq t_i \quad , \quad \Delta x = I_i(x,y),$

$\frac{dy}{dt} = \beta x + g(t,x,y) \quad , \quad t \neq t_i \quad , \quad \Delta y\Big|_{t = t_i} = G_i(x,y)$

where$x \in D_\lambda \subseteq R^n$, $y \in D_\beta \subseteq R^n$, D_λ is a closed and bounded domain.

The vector functions $f(t,x,y) and g(t,x,y)$are defined on the domain:

$(t,x,y) \in R^1 \times D_\lambda \times D_\beta = (-\infty,\infty) \times D_\lambda \times D_\beta$

Our work, we investigate the periodic solution for nonlinear system of differential equations of some operators with impulsive action, by using the numerical-analytic method for periodic solutions which is given by Samoilenko [5].

Consider the system of differential equations of some operators with impulsive action which has the form:

$$\left.\begin{aligned} \frac{dx}{dt} &= f(t,x,Ax,Bx), \tau \neq \tau_i \\ \Delta x|_{\tau=\tau_i} &= I_i(x,Ax,Bx) \end{aligned}\right\} \qquad (1)$$

where $x \in D$, D is the closure of bounded domain subset in R^n.
The vector functions $f(t,x,y,z)$ and $g(t,x,y)$ are defined on the domain
$(t,x,(y,z)) \in R^1 \times D \times D_1 = (-\infty,\infty) \times D \times D_1$ (2)
which are piecewise continuous functions in t,x,y,z and periodic in t of period T.
Let $I_i(x,y,z)$ be continuous vector functions, defined on the domain (2) and
$I_{i+p}(x,y,z) = I_i(x,y,z)$, $\tau_{i+p} - \tau_i = T$ (3)
for all $i \in z^+, x \in D, (z,y) \in D_1$ and for some number p, $\{\tau_i\}$is finite positive sequence of numbers.
Suppose that the operators A and B transform any piecewise continuous functions from the domain D to the piecewise contiuous function in the domain D_1 respectively. Moreover
$Ax(t+T) = Ax(t)$ and $Bx(t+T) = Bx(t)$.
Let the functions $f(t,x,y,z)$, $g(t,x,y,z)$, $I_i(t,x,y,z)$ and the operators A, B satisfy the following inequalities.
$\|f(t,x,y,z)\| \le M_1$, $\|I_i(t,x,y,z)\| \le M_2$, (4)
$\|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\| \le K_1\|x_1 - x_2\| + K_2\|y_1 - y_2\| + K_3\|z_1 - z_2\|$
$\|g(t,x_1,y_1,z_1) - g(t,x_2,y_2,z_2)\| \le Q_1\|x_1 - x_2\| + Q_2\|y_1 - y_2\| + Q_3\|z_1 - z_2\|$ (5)
and
$\|I_i(t,x_1,y_1,z_1,w_1) - I_i(t,x_2,y_2,z_2,w_2)\| \le L_1\|x_1 - x_2\| + L_2\|y_1 - y_2\| + L_3\|z_1 - z_2\| + \|w_1 - w_2\|$ (6)
$\|Ax_1(t) - Ax_2(t)\| + \|Bx_1(t) - Bx_2(t)\| \le (G_1 + G_2)\,\|x_1(t) - x_2(t)\|$ (7)
for all $t \in R^1$, $x, x_1, x_2 \in D, y, y_1, y_2, z, z_1, z_2 \in D_1$ where $M_1, M_2, K_1, K_2, K_3, Q_1, Q_2, Q_3, L_1, L_2, L_3$ and G_1, G_2 are a positive constants.
Consider the matrix

$$\Omega = \begin{pmatrix} K\frac{T}{3} & K \\ pTH & 2pH \end{pmatrix} \quad (8)$$

Where

$$K = (K_1 + K_2 + K_3)\big[1 + (G_1 + G_2) + (Q_1 + Q_2 + Q_3)\big(1 + (G_1 + G_2)\big)\big]$$

and

$$H = (L_1 + L_2 + L_3)[1 + (G_1 + G_2) + (Q_1 + Q_2 + Q_3)(1 + (G_1 + G_2))]$$

We define the non-empty sets as follows

$$\left.\begin{aligned} D_f &= D - M_1\frac{T}{2} + 2p\ M_2 \\ D_{1f} &= D_1 - [\,M_1\frac{T}{2} + 2p\ M_2](G_1 + G_2) \end{aligned}\right\} \quad (9)$$

Furthermore, we suppose that the greatest Eigen-value λ_{max} of the matrix Λ does not exceed unity, i.e.

$$\omega = K\frac{T}{2} + pH(2 + K\frac{T}{2}) < 1 \ . \quad (10)$$

Lemma 1. Let $f(t)$ be a continuous (piecewise continuous) vector function in the interval $0 \le t \le T$. Then

$$\left\| \int_0^t \left(f(s) - \frac{1}{T}\int_0^T f(s)ds\right)ds \right\| \le \alpha(t) \max_{t\in[0,T]} \|f(t)\|,$$

where $\alpha(t) = 2t(1 - \frac{t}{T})$. (For the proof see [5]) .

2. APPROXIMATE SOLUTION

The investigation of a periodic approximate solution of the system (1) makes essential use of the statements and estimates given below.

Theorem 1. If the system of integro-differential equations with impulsive action (1) satisfy the inequalities (3) to 7) and the conditions (8), (10) has a periodic solution $x = x(t,x_0)$, passing through the point $(0,x_0), x_0 \in D_f$, $Ax_0 \in D_{1f}$ and $Bx_0 \in D_{2f}$, then the sequence of functions:

$$x_{m+1}(t,x_0) = x_0 + \int_0^t [f(s,x_m(s,x_0),y_m(s,x_0),z_m(s,x_0))$$
$$-\frac{1}{T}\int_0^T g\big(s,x_m(s,x_0),y_m(s,x_0),z_m(s,x_0)\big)\,ds]ds$$
$$+ \sum_{0<\tau_i<t} I_i\left(x_m(t_i,x_0),y_m(t_i,x_0),z_m(t_i,x_0)\right)$$

$-\frac{t}{T}\sum_{i=1}^{p} I_i(x_m(t_i,x_0), y_m(t_i,x_0), z_m(t_i,x_0)),$ (11)

with

$x_0(t,x_0) = x_0$ and $Ax_m(t,x_0) = y_m(t,x_0)$ and $Bx_m(t,x_0) = z_m(t,x_0)$,

$m = 0,1,2,\cdots$

is periodic in t of period T, and uniformly convergent as $m \to \infty$ in

$(t,x_0) \in R^1 \times D_f = (-\infty,\infty) \times D_f$ (12)

to the vector function $x^0(t,x_0)$ defined on the domain (12), which is periodic in t of period T and satisfying the system of integral equation

$$x(t,x_0) = x_0 + \int_0^t [f(s,x(s,x_0),y(s,x_0),z(s,x_0))$$
$$-\frac{1}{T}\int_0^T g(s,x(s,x_0),y(s,x_0),z(s,x_0))\,ds]ds$$
$$+\sum_{0<\tau_i<t} I_i\left(x(t_i,x_0),y(t_i,x_0),z(t_i,x_0)\right)$$
$$-\frac{t}{T}\sum_{i=1}^{p} I_i\left(x(t_i,x_0),y(t_i,x_0),z(t_i,x_0)\right), \quad (13)$$

which a unique solution of the system (1) provided that

$$\|x^0(t,x_0) - x_0\| \le M_1\frac{T}{2} + 2p\ M_2 \quad (14)$$

and

$$\|x^0(t,x_0) - x_m(t,x_0)\| \le \omega^m(1-\omega)^{-1}(M_1\frac{T}{2} + 2p\ M_2) \quad (15)$$

for all $m \ge 1, t \in R^1$, where the eigen-value λ of the matrix Ω is a positive fraction less than one.

Proof. Consider the sequence of functions $x_1(t,x_0), x_2(t,x_0),\cdots,x_m(t,x_0),\cdots$,

defined by recurrence relation (11). Each of the functions of the sequence is periodic in t of period T.

Now, by Lemma 1, we have

$$\|x_m(t,x_0) - x_0\| = \|x_0 + \int_0^t [f(s,x_m(s,x_0),y_m(s,x_0),z_m(s,x_0)) -$$
$$-\frac{1}{T}\int_0^T g(s,x_m(s,x_0),y_m(s,x_0),z_m(s,x_0))\,ds]ds +$$
$$+\sum_{0<\tau_i<t} I_i\,(x_m(t_i,x_0),y_m(t_i,x_0),z_m(t_i,x_0)) -$$
$$-\frac{t}{T}\sum_{i=1}^{p} I_i\left(x_m(t_i,x_0),y_m(t_i,x_0),z_m(t_i,x_0)\right) - x_0\|$$
$$\le (1-\frac{t}{T})\int_0^t \|f(s,x_m(s,x_0),y_m(s,x_0),z_m(s,x_0))\|\,ds +$$
$$+\frac{t}{T}\int_t^T \|f(s,x_m(s,x_0),y_m(s,x_0),z_m(s,x_0))\|ds + + \sum_{0<\tau_i<t} I_i\,\|(x_m(t_i,x_0),y_m(t_i,x_0))\|$$
$$+\frac{\|t\|}{T}\sum_{i=1}^{p}\left\|I_i\left(x_m(t_i,x_0),y_m(t_i,x_0)\right)\right\| \le M_1\frac{T}{2} + 2p\ M_2 \quad (16)$$

From (7) and (14), we have

$$\|Ax_m(t,x_0) - Ax_0\| + \|Bx_m(t,x_0) - Bx_0\| \le [M_1\frac{T}{2} + 2p\ M_2](G_1 + G_2) \quad (17)$$

for all $x_0 \in D_f$.

For $m = 1$, in (2.1), we get

$$\|x_2(t,x_0) - x_1(t,x_0)\| \le (1-\frac{t}{T})\int_0^t K_1\|x_1(s,x_0) - x_0\| + K_2\|y_1(s,x_0) - y_0(s,x_0)\|$$
$$+ K_3\|z_1(s,x_0) - z_0(s,x_0)\|\,)ds +$$

$$+\frac{t}{T}\int_{t}^{T} K_1\|x_1(s,x_0)-x_0\| + K_2\|y_1(s,x_0)-y_0(s,x_0)\|$$
$$+K_3\|z_1(s,x_0)-z_0(s,x_0)\|\,)ds +$$
$$+\sum_{0<\tau_i<t} L_1\|x_1(t_i,x_0)-x_0\| + L_2\|y_1(t_i,x_0)-y_0(t_i,x_0)\|$$
$$+L_3\|z_1(t_i,x_0)-z_0(t_i,x_0)\| + \|w_1(t_i,x_0)-w_0(t_i,x_0)\|)$$
$$+\frac{\|t\|}{T}\sum_{i=1}^{p} L_1\|x_1(t_i,x_0)-x_0\| + L_2\|y_1(t_i,x_0)-y_0(t_i,x_0)\|$$
$$+L_3\|z_1(t_i,x_0)-z_0(t_i,x_0)\|\,)$$
$$\le\left(1-\frac{t}{T}\right)\int_0^t K\left(M_1\frac{T}{2}+2p\ M_2\right)ds$$
$$+\frac{t}{T}\int_t^T K(M_1\frac{T}{2}+2p\ M_2)ds + H(M_1\frac{T}{2}+2p\ M_2)(G_1+G_2)$$
$$\le\alpha(t)(M_1\frac{T}{2}+2p\ M_2)+H(M_1\frac{T}{2}+2p\ M_2)(G_1+G_2)$$
$$\le F_1\alpha(t)+F_2=F_1\frac{T}{2}+F_2 \tag{18}$$

Where

$$F_1 = K(M_1\frac{T}{2}+2p\ M_2)$$

and

$$F_2 = H(M_1\frac{T}{2}+2p\ M_2)(G_1+G_2)$$

If the following inequality is true

$$\|x_m(t,x_0)-x_{m-1}(t,x_0)\| \le F_{1_{(m-1)}}\alpha(t)+F_{2_{(m-1)}}$$
$$\le F_{1_{(m-1)}}\frac{T}{2}+F_{2_{(m-1)}} \tag{19}$$

for all $m = 1,2,\cdots$.

Then, we shall to prove that

$$\|x_{m+1}(t,x_0)-x_m(t,x_0)\| \le K(F_{1_{(m-1)}}\frac{T}{2}+F_{2_{(m-1)}})\alpha(t)+2pH(F_{1_{(m-1)}}\frac{T}{2}+F_{2_{(m-1)}}) \tag{20}$$

for all $m = 0,1,2,\cdots$.

By mathematical induction, we have

$$\|x_{m+1}(t,x_0)-x_m(t,x_0)\| \le F_{1_{(m)}}\alpha(t)+F_{2_{(m)}} \le \frac{T}{2}F_{1_{(m)}}+F_{2_{(m)}}, \tag{21}$$

where

$$F_{1_{(m+1)}} = K\frac{T}{2}F_{1_{(m)}}+KF_{2_{(m)}},\ F_{2_{(m+1)}} = pHTF_{1_{(m)}}+2pHF_{2_{(m)}}, \tag{22}$$
$$F_{1_{(0)}} = (M_1\frac{T}{2}+2p\ M_2),\ \ F_{2_{(0)}} = 2p(M_1\frac{T}{2}+2p\ M_2),$$
$$m = 0,1,2,\cdots.$$

It is sufficient to show that all solutions of (11) approach zero as $m\to\infty$, i.e. it is necessary and sufficient that the eigen values of the matrix Λ are assumed to be within a unit circle.

It is well-known that the characteristic equation of the matrix Λ is

$$K\frac{T}{2}+pH\left(2+K\frac{T}{2}\right)<1 \tag{23}$$

And this ensures that the sequence of functions (11) is convergent uniformly on the domain (12) as $m\to\infty$.

Let

$$\lim_{m\to\infty} x_m(t,x_0) = x_\infty(t,x_0), \tag{24}$$

Since the sequence of functions (11) is periodic in t of period T, then the limiting is also periodic in t of period T.

Moreover, By lemma 1 and (24) and the following inequality

$$\|x_{m+1}(t,x_0)-x_m(t,x_0)\| \le \sum_{i=0}^{k-1}\|x_{m+i+1}(t,x_0)-x_{m+i}(t,x_0)\| \le$$

$$\leq \sum_{i=0}^{k-1} \omega^{m+i} \left(M_1 \frac{T}{2} + 2p\ M_2 \right)$$

is hold and the inequalities (14) and (15) are satisfied for all $m \geq 0$.

Finally, we have to show that $x(t, x_0)$ is unique solution of (1). on the contrary, we suppose that there is at least two different solutions $x(t, x_0)$ and $r(t, x_0)$ of (1).

From (13), the following inequality holds:

$$\|x(t, x_0) - r(t, x_0)\| \leq \left(1 - \frac{t}{T}\right) \int_0^t K \|x(s, x_0) - r(s, x_0)\| ds$$

$$+ \frac{t}{T} \int_t^T K\|x(s, x_0) - r(s, x_0)\|\, ds + \sum_{0<\tau_i<t} H \|x(t_i, x_0) - r(t_i, x_0)\|$$

$$+ \frac{t}{T} \sum_{i=1}^{p} H \|x(t_i, x_0) - r(t_i, x_0)\| \tag{25}$$

Setting $\|x(t, x_0) - r(t, x_0)\| = h(t)$, the inequality (25) can be written as:

$$h(t) \leq (1 - \frac{t}{T}) \int_0^t K\, h(s) ds + \frac{t}{T} \int_t^T Kh(s)\, ds + \sum_{0<\tau_i<t} H\, h(t_i) + \frac{t}{T} \sum_{i=1}^{p} H\, h(t_i)$$

Let $\max_{t\in[0,T]} h(t) = h_0 \geq 0$. By iteration, we get:

$$h(t) \leq N_m \alpha(t) + M_m \tag{26}$$

From (2.14), we have:

$$\begin{pmatrix} N_{m+1} \\ M_{m+1} \end{pmatrix} = \begin{pmatrix} K\frac{T}{2} & K \\ pTH & 2pH \end{pmatrix} \begin{pmatrix} N_m \\ M_m \end{pmatrix}, \tag{27}$$

which satisfies the initial conditions $N_0 = 0, M_0 = h_0$that is

$$\begin{pmatrix} N_m \\ M_m \end{pmatrix} = \begin{pmatrix} K\frac{T}{2} & K \\ pTH & 2pH \end{pmatrix}^m \begin{pmatrix} 0 \\ h_m \end{pmatrix}. \tag{28}$$

Hence it is clear that if the condition (23) is satisfied then $N_m \to 0$ and $M_m \to 0$ as $m \to \infty$.

From the relation (26) we get $h(t) \equiv 0$ or $x(t, x_0) = r(t, x_0)$, i.e. $x(t, x_0)$ is a unique solution of (1). ∎

3. EXISTENCE OF SOLUTION

The problem of the existence of periodic solution of period T of the system (1) is uniquely connected with the existence of zeros of the function $\Delta(x_0)$ which has the form:

$$\Delta(x_0) = \frac{1}{T} [\int_0^T f(t, x^0(t, x_0), y^0(t, x_0), z^0(t, x_0))\, dt\ + \sum_{i=1}^{p} I_i\, (x^0(t_i, x_0), y^0(t_i, x_0), z^0(t_i, x_0)\,)] \tag{29}$$

Since this function is approximately determined from the sequence of functions

$$\Delta_m(x_0) = \frac{1}{T} [\int_0^T f(t, x_m(t, x_0), y_m(t, x_0), z_m(t, x_0))\, dt$$

$$+ \sum_{i=1}^{p} I_i\, (x_m(t_i, x_0), y_m(t_i, x_0), z_m(t_i, x_0)\,)] \tag{30}$$

where$x^0(t, x_0)$ is the limiting of the sequence of functions (11). Also

$y^0(t, x_0) = Ax^0(t, x_0)$ and $z^0(t, x_0) = Bx^0(t, x_0)$

Now, we prove the following theorem taking the following inequality will be satisfied for all $m \geq 1$.

$$\|\Delta(x_0) - \Delta_m(x_0)\| \leq\ \omega^m (1 - \omega)^{-1} (K + \frac{p}{T} H)(M_1 \frac{T}{2} + 2p\ M_2) \tag{31}$$

Theorem 2. If the system of equations (1) satisfies the following conditions:

(i)The sequence of functions (11) has an isolated singular point $x_0 = x^0$, $\Delta_m(x_0) \equiv 0$, for all $t \in R^1$;

(ii)The index of this point is nonzero;

(iii)There exists a closed convex domain D_4 belonging to the domain D_f and possessing a unique singular point x^0 such that on it is boundary Γ_{D_4} the following inequality holds

$$\inf_{x\in\Gamma_{D_2}} \|\Delta_m(x^0)\| \leq \omega^m(1-\omega)^{-1}(K+\frac{p}{T}H)(M_1\frac{T}{2}+2p\ M_2)$$

for all $m \geq 1$. Then the system (1) has a periodic solution $x = x(t, x_0)$ for which $x(0) \in D_2$.

Proof. By using the inequality (31) we can prove the theorem is a similar way to that of theorem 2.1.2 [2].

Remark 3.1[5]. When $R^n = R^1$, i.e. when x^0 is a scalar, theorem 3.1 can be proved by the following.

Theorem 3.2. Let the functions $f(t, x, y, y, z, w)$ and $I_i(x, y, z, w)$ of system (11) are defined on the interval $[a, b]$ in R^1. Then the function (30) satisfies the inequalities:

$$\left.\begin{aligned} \min_{a+(M_1\frac{T}{2}+2p\ M_2)\leq x^0\leq b-(M_1\frac{T}{2}+2p\ M_2)} \Delta_m(x^0) \leq -\,\omega^m(1-\omega)^{-1}(K+\tfrac{p}{T}H) \\ \max_{a+(M_1\frac{T}{2}+2p\ M_2)\leq x^0\leq b-(M_1\frac{T}{2}+2p\ M_2)} \Delta_m(x^0) \geq \omega^m(1-\omega)^{-1}(K+\tfrac{p}{T}H) \end{aligned}\right\} \quad (32)$$

Then (1.1) has a periodic solution in t of period T for which

$$x^0(0) \in [a + (M_1\frac{T}{2} + 2p\ M_2), b - (M_1\frac{T}{2} + 2p\ M_2)]\,.$$

Proof. Let x_1 and x_2 be any points of the interval [a,b] such that

$$\left.\begin{aligned} \Delta_m(x_1) = \min\ \Delta_m(x^0)_{a+(M_1\frac{T}{2}+2p\ M_2)\leq x^0\leq b-(M_1\frac{T}{2}+2p\ M_2)}, \\ \Delta_m(x_2) = \max_{a+(M_1\frac{T}{2}+2p\ M_2)\leq x^0\leq b-(M_1\frac{T}{2}+2p\ M_2)} \Delta_m(x^0)\ . \end{aligned}\right\} \quad (33)$$

By using the inequalities (31) and (32) , we have

$$\left.\begin{aligned} \Delta_m(x_1) = \Delta_m(x_1) + \left(\Delta_m(x_1) - \Delta_m(x_1)\right) < 0\ , \\ \Delta_m(x_2) = \Delta_m(x_2) + (\Delta_m(x_2) - \Delta_m(x_2)) > 0\ . \end{aligned}\right\} \quad (34)$$

From the continuity of $\Delta(x^0)$ and by using (3.6), there exists a point $x^0, x^0 \in [x_1, x_2]$, such that $\Delta(x^0) \equiv 0$, i.e. $x = x(t, x_0)$ is a periodic solution in t of period T for which

$$x^0(0) \in [a + (M_1\frac{T}{2} + 2p\ M_2), b - (M_1\frac{T}{2} + 2p\ M_2)]\,.$$

Remark 3.2. It is clear that when we put $I_i \equiv 0$, we get a periodic solution of (1) without introducing impulsive action.

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