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European Journal of Advances in Engineering and Technology, 2019, 6(2):42-53

Research Article ISSN: 2394 - 658X

Visual Models and Classification of 1D, 2D, and 3D Random Walk

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ABSTRACT

A new visual model has been proposed for describing 1D, 2D and 2D random walk based on consideration of linear and nonlinear arithmetic triangles (1D), linear and nonlinear arithmetic squarer (2D), and linear and nonlinear arithmetic octahedrons (3D Our studies and visual constructions presented in this work show that the nonlinear case of an arithmetic triangle coincides with the optical scheme of the laser. Geometric constructions and recursive formulas are given. The studies presented in this paper show various geometric properties and nonlinear effects of 1D, 2D, and 3D spaces. For nonlinear 1D and 3D cases we can speak of filling the numbers of the arithmetic triangle (1D) and the arithmetic octahedron (3D) in the form of "islands of numbers" or separate structures of numbers.

Key words: *deterministic model, arithmetic figures, geometrization of physics, random walk, nonlinear effects*

1. INTRODUCTION

Pascal's arithmetic triangle has been known since ancient times [1]. In [2 – 9] we carried out studies of Pascal's triangle, its analogues, generalizations, and possible applications of this visual geometric model. In [10] we proposed a new, stepwise form for Pascal's triangle (1D) and its two-sided (2D) generalization. In [11 sticks] we proposed an illustrative algorithm for describing 2D generalization of Pascal's triangle using counting sticks. In [12] various multidimensional generalizations of Pascal's arithmetic triangle were considered.

1D an 2D random were shown in [13, 14]. The 3D case of random walk was shown in [13 wiki random]. In [15] we proposed an optical laser scheme that is a nonlinear 1D walk in a system of rays; in a real laser, nonlinear 2D random walk in a system of rays was actually carried out. Nonlinear and non-Markovian random walk was described in [16].

In [17] we proposed the classification of visual linear and nonlinear 1D and 2D models in a form of the arithmetic triangles and squares to describe linear and nonlinear 1D and 2D random walk. In [18] a symmetric random walk on a regular tetrahedron, octahedron, and a cube was described. In [19] we proposed visual linear and nonlinear 3D models in a form of the arithmetic octahedrons and respectively to describe linear and nonlinear 3D random walk.

In the 1D case (along a straight line) [13, 14, 17] a random walk (linear and nonlinear) can occur along two mutually perpendicular directions (right, left) inside an arithmetic triangle [17] (the triangle has two corners on his base and one on his top).

In the 2D case [13, 14, 17] a random walk (linear and nonlinear) can be carried out in four different directions (forward, back, right, left) inside an arithmetic square [17] (the square has four corners).

In the 3D case [19] a random walk (linear and nonlinear) can be carried out in six (forward, backward, right, left, up, down) different directions inside an octahedron (the octahedron has six vertices).

In this paper, we combine, process, and complement our two studies on visual 1D, 2D [17] and 3D [19] linear and nonlinear random walk models.

In our studies, we adhere to the approach proposed by Felix Klein about the need to create simple and visual geometric models to substantiate the consistency in describing complex phenomena [20, 21].

2. 1D Random Walk

For the convenience of the reader we first describe the well-known linear random walk.

2.1. Linear random 1D walk in the triangle

A 1D linear random walk (or walk along a straight line [13, 14, 17]) can be described using the usual Pascal triangle [13, 14, 17]. Figure 1 shows the usual form of Pascal's triangle constructed using oblique lines:

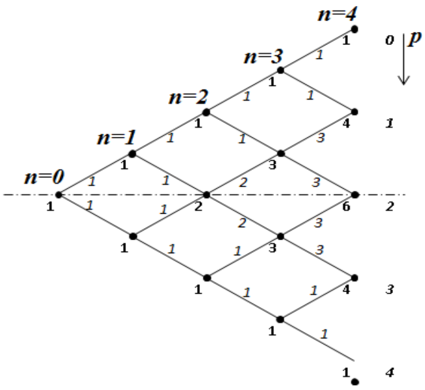

Fig. 1

Figure 1. The first five rows of the Pascal triangle constructed with the help of oblique lines. The rays on this linear pattern have the same length and are inclined at the same angles. After each pass (iteration) the rays are split in two, some of the rays overlap each other. The number of rays is indicated by oblique numbers. The number of rays arriving at the branch points is shown in bold numbers, which in turn are linear binomial coefficients.

Figure 2 shows another form of Pascal's triangle constructed using small squares:

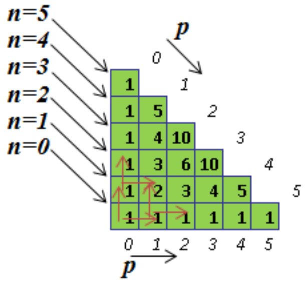

Fig. 2

Figure 2. The first six rows of the Pascal triangle constructed with the help of small squares. The red arrows of the same length and perpendicular to each other show the two directions of summation of the binomial coefficients. We obtain the binomial coefficients of the series n using arrows of unit length. It is a diagram of a linear random 1D walk [14, 17].
The rows in the triangle in this example are denoted by n: $n = 0, 1, 2, \dots$, the numbers in the row are denoted by p: $p = 0, 1, 2, \dots, n$.

The numbers in the Pascal triangle are linear binomial coefficients $\binom{n}{p}$; they can be found [14, 17] using the recursive expression:

$$\binom{n}{p} = \binom{n-1}{p-1} + \binom{n-1}{p}. \qquad (1)$$

It is necessary to specify the numbers of the zero row ($n = 0$) or in other words the initial conditions:

$$\binom{0}{p} = 1 \text{ for } p = 0 \text{ and } \binom{0}{p} = 0 \qquad (2)$$

for other values of p.
Examples of calculations using formula (1) can be checked directly in Figure 2.
Figure 3 shows another way to obtain [7] binomial coefficients using a table of numbers in binary numbering system consisting of 0 and 1:

n=3

	1	1	1	1	0	0	0	0
	1	1	0	0	1	1	0	0
	1	0	1	0	1	0	1	0
Σ	**3**	**2**	**2**	**1**	**2**	**1**	**1**	**0**
binomial → cotfficients	*1*	*3*		*3*				*1*

Fig. 3

Figure 3. The vertical columns of the table contain numbers in binary numbering systems: 111, 110, 101, etc. The sum of these numbers is presented in the first row below the table: 3, 2, 2, etc. The second row under the table shows the repeatability of these sums: the number 3 occurs once, the number 2 occurs three times, and so on; these numbers are linear binomial coefficients.
In Figure 2. It can be seen that the numbers densely (without spaces) fill the arithmetic triangle.

2.2. Nonlinear random 1D walk in the triangle

A nonlinear analogue of Pascal's triangle was proposed in [15]. This optical design is shown in Figure 6:

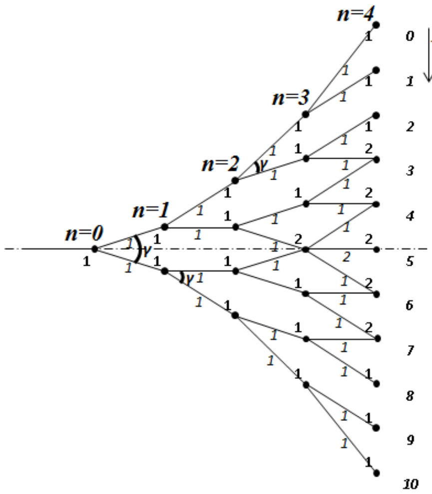

Fig. 4

Figure 4. The nonlinear analogue of Pascal's triangle. The rays in this scheme have the same length and are inclined at small angles: $\gamma \approx sin\gamma, 2\gamma \approx sin2\gamma, 3\gamma \approx sin3\gamma, \dots$. After each pass (iteration) the rays are split in two, some of the rays overlap each other. The numbers of rays are indicated by oblique numbers. The numbers of rays arriving at the branch points are shown in bold numbers which in turn are non-linear binomial coefficients.

Figure 5 shows another form of the nonlinear analogue of Pascal's triangle constructed using small squares:

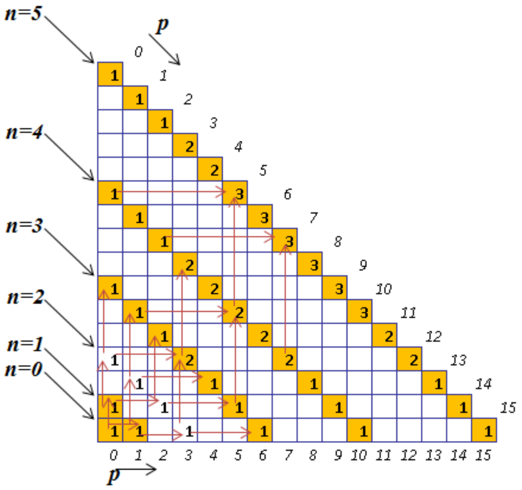

Figure 5. The first six rows of the nonlinear analogue of Pascal triangle constructed with the help of small squares. The red arrows of different lengths and perpendicular to each other show the two directions of summation of the nonlinear binomial coefficients. The nonlinear binomial coefficients of the series $n = 1$ are obtained with the help of two arrows of unit length; nonlinear binomial coefficients of the series $n = 2$ we get with the help of arrows of length 2; nonlinear binomial coefficients of the series $n = 3$ we get using arrows of length 3; etc. It is a diagram of a nonlinear random 1D walk.
The rows in the nonlinear triangle [7, 17] in this example are denoted by n: $n = 0, 1, 2, \dots$, the numbers in the row are denoted by p: $p = 0,1, 2, \dots, n(n + 1)/2$.
The numbers in the nonlinear triangle are nonlinear binomial coefficients $\binom{n}{p}$ they can be found [7, 17] using the recursive nonlinear expression (similar to expression (1)):

$$\binom{n}{p} = \binom{n-1}{p-n} + \binom{n-1}{p}. \tag{3}$$

It is necessary to specify the numbers of the zero row $(n = 0)$ or in other words the initial conditions:

$$\binom{0}{p} = 1 \text{ for } p = 0 \text{ and } \binom{0}{p} = 0 \tag{4}$$

for other values of p.

Examples of calculations using formula (4) can be checked directly in Figure 5.

Figure 6 shows another way [7, 17] to obtain non-linear binomial coefficients using a table of numbers in a quasi-binary numbering system consisting of 0, 1, 2, 3, and so on:

n=3

	3	3	3	3	0	0	0	0
	2	2	0	0	2	2	0	0
	1	0	1	0	1	0	1	0
Σ	6	5	4	3	3	2	1	0
binomial cotfficients →	*1*	*1*	*1*	*2*		*1*	*1*	*1*

Fig. 6

Figure 6. The vertical columns of the table contain numbers in the quasi-binary numbering system: 321, 320, 301, etc. The numbers 2 and 3 are put instead of ones in the usual binary numbering system and correspond to the number position of the numbers in the usual binary numbering system. The sum of these numbers is presented in the first row below the table: 6, 5, 4, etc. The second row below the table shows the repeatability of these sums: the number 6 occurs once, the number 5 occurs once, the number 3 meets twice, etc.; these numbers are non-linear binomial coefficients.

In Figure 5. it can be seen that the numbers loosely (with spaces) fill in the arithmetic triangle, “islands of numbers” or separate structures of numbers in the form of inclined lines are formed.

3. 2D Random Walk

3.1. Linear random 2D walk in the square

The 2D linear random walk on the plane [14, 17] can be described in linear square using the 2D generalization of linear Pascal's triangle (Figure 2):

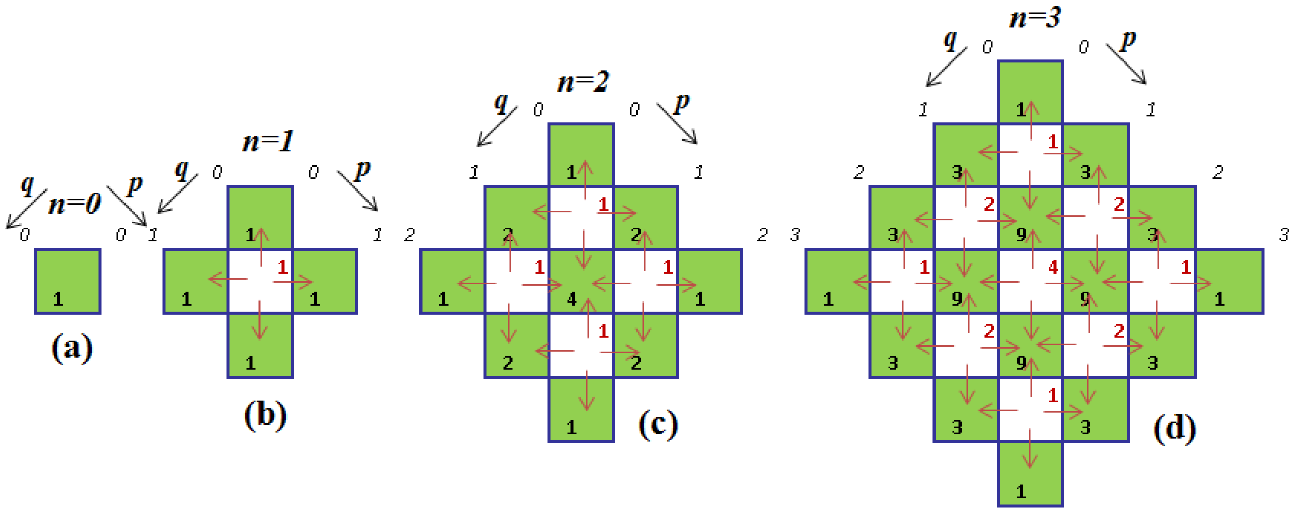

Figure 7. Diagram of a linear random 2D walk [14, 17]. Red arrows of the same length and with different four directions show the direction of summation of the binomial coefficients. We obtain the binomial coefficients of the series (matrices) n using arrows of unit length. The previous iteration is shown by red numbers in the upper right inside the small squares.

Let's rotate the squares shown in Figure 7 by 45 ° and write down successively the numbers of 2D generalization of Pascal's triangle [11, 17] in the form of the corresponding tables (matrices):

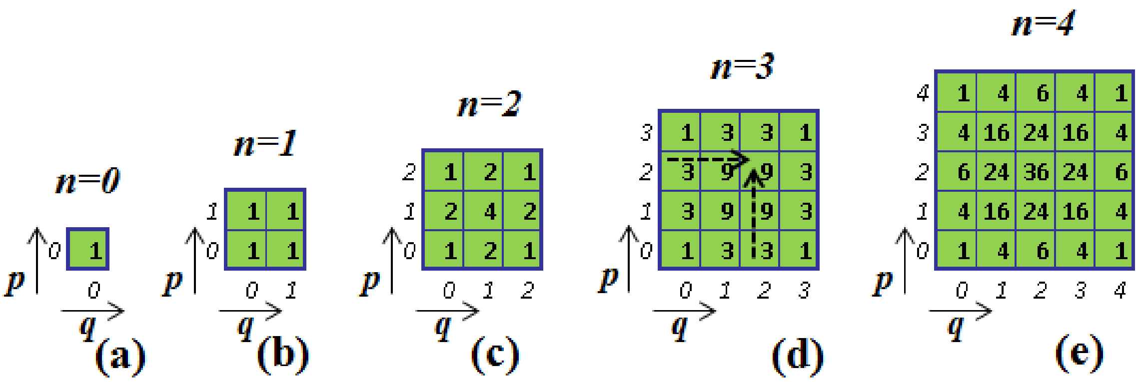

Figure 8. Numbers corresponding to the first five rows (consisting of five matrices n) of a generalized 2D Pascal's triangle. The usual linear binomial coefficients (1D) are located along the sides of the squares. The numbers inside squares are products of intersection (shown by dotted arrows) of two corresponding binomial coefficients [11, 17].
The rows (matrices) in 2D case in this example are denoted by n: $n = 0, 1, 2, \dots$, the numbers in the matrix are denoted by p and q: $p = 0,1, 2, \dots, n$; $q = 0,1, 2, \dots, n$.

Denote a number located in the n – matrix (n – row) as $\begin{pmatrix} n \\ p \\ q \end{pmatrix}$ then we specify the numbers of the zero row ($n = 0$) or the initial conditions:

$$\begin{pmatrix} 0 \\ p \\ q \end{pmatrix} = 1 \text{ for } p = 0, q = 0 \text{ and } \begin{pmatrix} 0 \\ p \\ q \end{pmatrix} = 0 \qquad (5)$$

for other values of p and q.

We can obtain an expression [11, 17] (similar to linear expression (1)) for the successive construction of our linear generalized triangle using square matrices for n- series:

$$\begin{pmatrix} n \\ p \\ q \end{pmatrix} = \begin{pmatrix} n-1 \\ p-1 \\ q-1 \end{pmatrix} + \begin{pmatrix} n-1 \\ p-1 \\ q \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q-1 \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q \end{pmatrix}. \qquad (6)$$

Examples of calculations using formula (4) can be checked directly in Figures 4 and 5.
In Figure 5. It can be seen that the numbers densely (without spaces) fill the arithmetic square.

3.2. Nonlinear random 2D walk in the square

The 2D nonlinear random walk [17] on a plane can be described in nonlinear square using the 2D generalization of nonlinear arithmetic triangle (Figure 5):

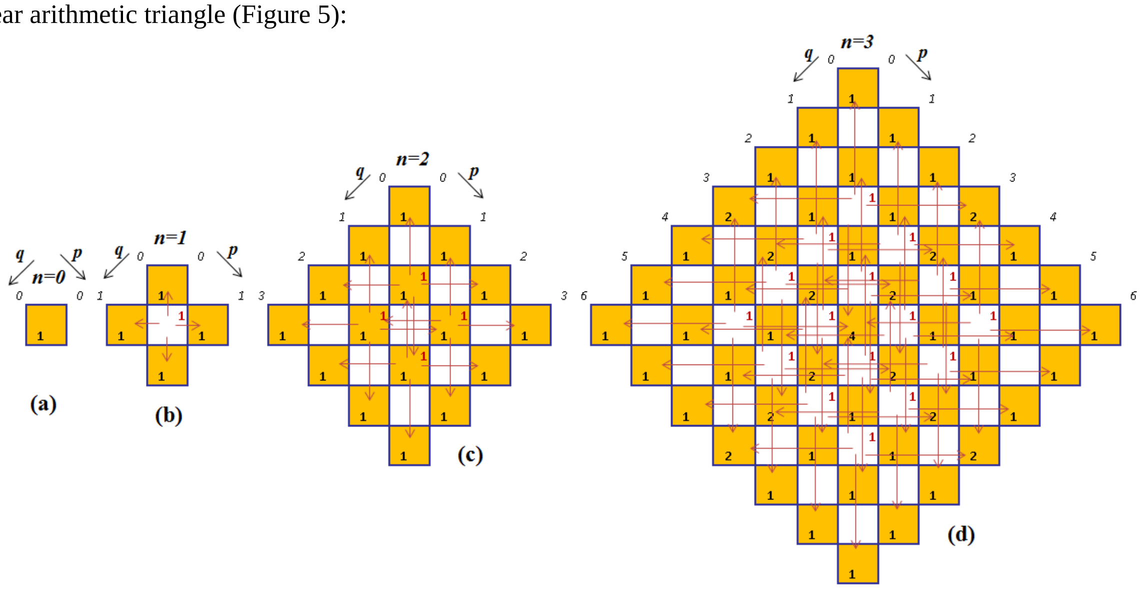

Figure 9. Diagram of a nonlinear random 2D walk. The red arrows of different lengths and different four directions show the direction of summation of the nonlinear binomial coefficients. The nonlinear binomial coefficients of the series (matrix) $n = 1$ we get with the help of four arrows of unit length; the nonlinear binomial coefficients of the series $n = 2$ are obtained using arrows of length 2; non-linear binomial coefficients of the series $n = 3$ we get using arrows of length 3; etc. The previous iteration is highlighted shown by red numbers in the upper right corner inside the small squares.

Let us rotate the squares depicted in Figure 8 by 45 °and write down successively the numbers of 2D generalization of nonlinear triangle in the form of the corresponding tables (matrices):

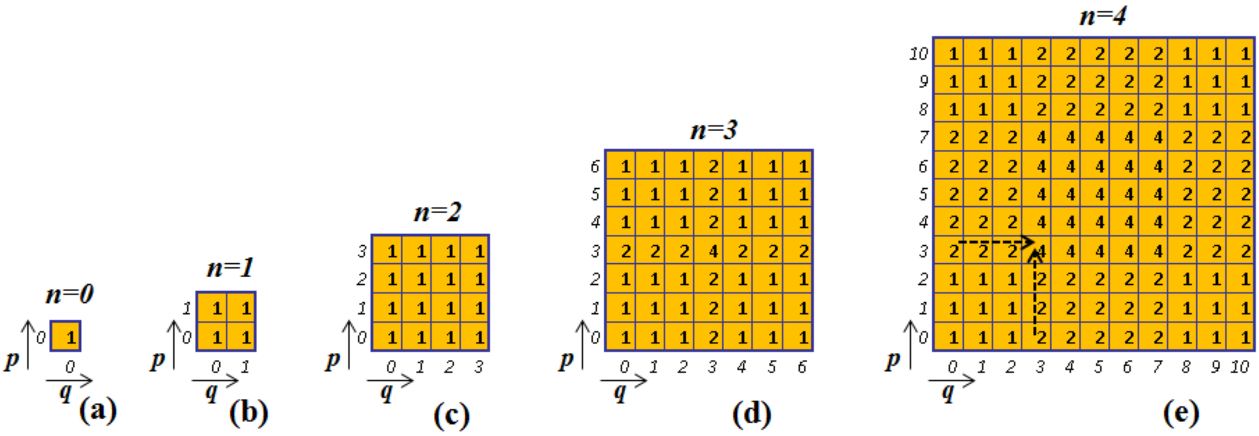

Figure 10. Numbers corresponding to the first five rows (consisting of five matrices n) of a generalized 2D nonlinear triangle. The nonlinear binomial coefficients (1D) are located along the sides of the squares. The numbers inside squares are products of intersection (shown by dotted arrows) of two corresponding binomial coefficients.

The rows (matrices) in 2D case in this nonlinear example are denoted by n: $n = 0, 1, 2, \dots$, the numbers in the matrix are denoted by p and q: $p = 0,1,2,\dots,n(n+1)/2$; $q = 0,1,2,\dots,n(n+1)/2$.

Denote a number located in the n - matrix) as $\begin{pmatrix} n \\ p \\ q \end{pmatrix}$ then we specify the numbers of the zero row ($n = 0$) or the initial conditions:

$$\begin{pmatrix} 0 \\ p \\ q \end{pmatrix} = 1 \text{ for } p = 0, q = 0 \text{ and } \begin{pmatrix} 0 \\ p \\ q \end{pmatrix} = 0 \qquad (7)$$

for other values of p and q.

We can obtain an expression (similar to expressions (3) and (6)) for the successive construction of our generalized nonlinear triangle (matrix) using square matrices for n- series:

$$\begin{pmatrix} n \\ p \\ q \end{pmatrix} = \begin{pmatrix} n-1 \\ p-n \\ q-n \end{pmatrix} + \begin{pmatrix} n-1 \\ p-n \\ q \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q-n \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q \end{pmatrix}. \qquad (8)$$

Examples of calculations using formula (8) can be checked directly in Figures 9 and 10.

In Figure 10. It can be seen that the numbers densely (without spaces) fill the arithmetic square.

4. 3D Random Walk

4.1. Linear random 3D walk in the octahedron

In paragraphs 2.1 and 3.1 we showed in details the summation of binomial coefficients for 1D and 2D cases using red arrows of the same length and different perpendicular directions. We obtained the binomial coefficients of the linear rows for 1D case by using two single-length arrows and we obtained the binomial coefficients of the linear squares for 2D case by using four single-length arrows.

In the proposed paper to describe a 3D random walk we do not draw arrows (as in the monograph [14]) due to space savings. You can calculate the linear coefficients using the recursion formula below and check it in the figures.

A 3D linear random walk or a walk in a volume can be described using a 3D model in the form of an arithmetic regular octahedron [19]. We obtain the linear binomial coefficients in the computed cell of the octahedron n by summing the numbers from the six cells adjacent to the computed cell.

Figures 11 - 14 show sequentially (for the first four iterations) images of arithmetic octahedrons composed of small cubes. (a) shows images of the arithmetic octahedrons themselves, (b) shows images of layers of octahedrons composed of small cubes containing numbers. These numbers correspond to the number of walks from the initial cell (initial cube) to the final cell (final cube).

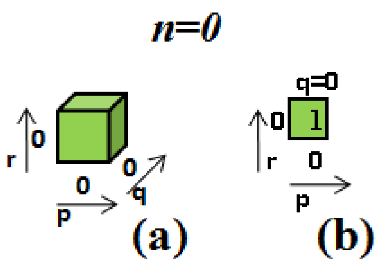

Fig. 11 The zero linear arithmetic octahedron (zero iteration $n = 0$) consists of 1 cube.

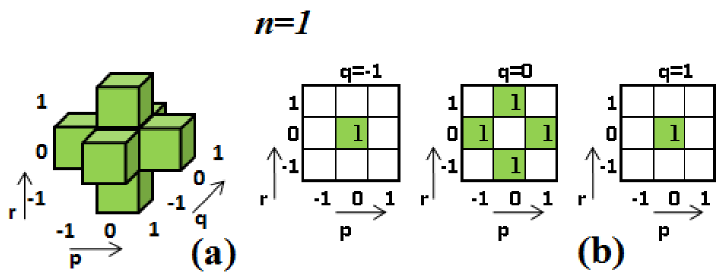

Fig.12 The first linear arithmetic octahedron (the first iteration $n = 1$) consists of 6 cubes.

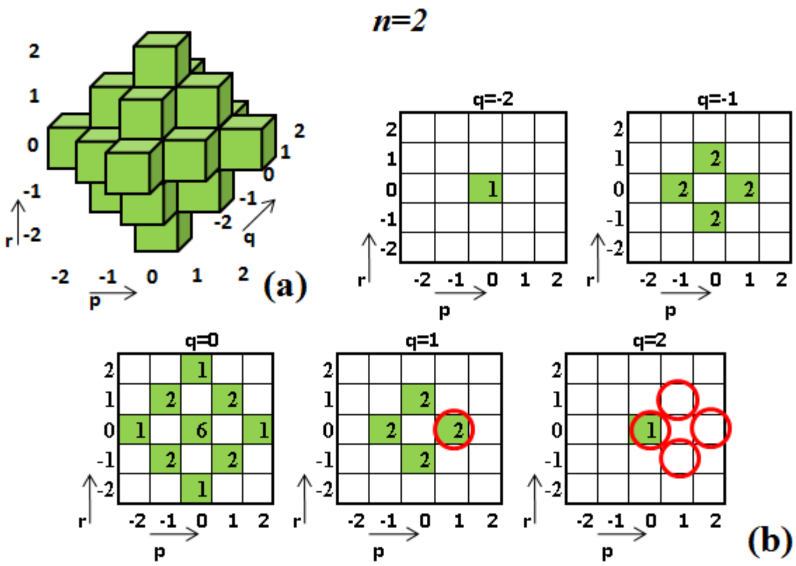

Fig. 13 The second linear arithmetic octahedron (second iteration $n = 2$) consists of 19 cubes.

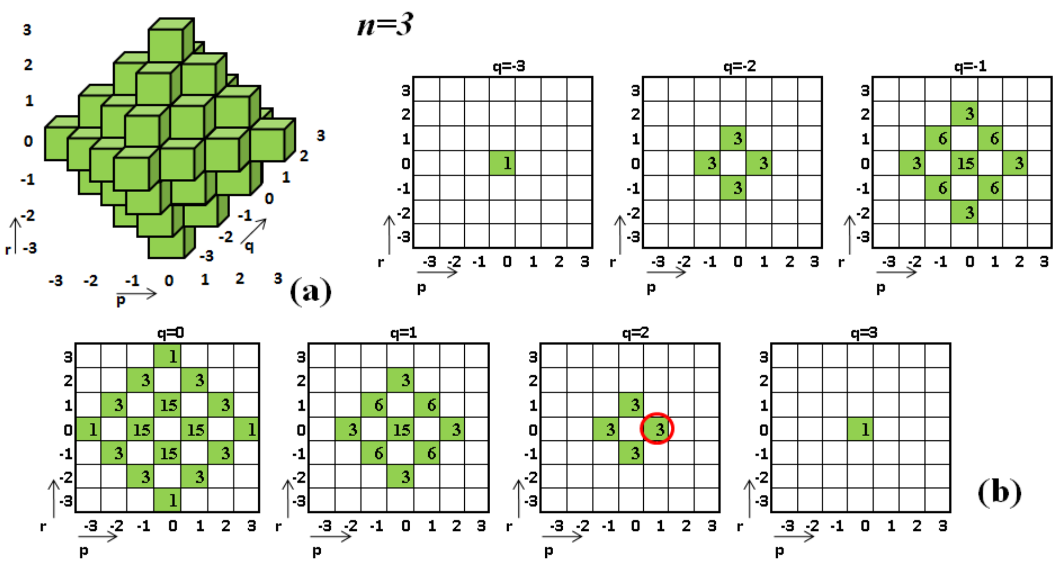

Fig. 14 The third linear arithmetic octahedron (the third iteration $n = 3$) consists of 44 cubes.

The sequence of numbers of octahedrons for the 3D case in this example is denoted by n: $n = 0, 1, 2, ...$ Total sum of numbers in octahedrons is 6^n. The numbers characterizing the octahedron (the numbers characterizing the position of the cubes of which the octahedron is composed) are denoted p, q and r:

$$\left.\begin{array}{r} p = 0, \pm 1, \pm 2, \dots, \pm n; \\ q = 0, \pm 1, \pm 2, \dots, \pm n: \\ r = 0, \pm 1, \pm 2, \dots, \pm n. \end{array}\right\} \tag{9}$$

Denote a number located in the n – octahedron as $\begin{pmatrix} n \\ p \\ q \\ r \end{pmatrix}$ then specify the number of the zero octahedron ($n = 0$) or in other words the initial conditions:

$$\begin{pmatrix} 0 \\ p \\ q \\ r \end{pmatrix} = 1 \text{ for } p = 0, q = 0, r = 0 \text{ and } \begin{pmatrix} 0 \\ p \\ q \\ r \end{pmatrix} = 0 \tag{10}$$

for the other values p, q and r.

.

We can get an expression for the consistent construction of our linear arithmetic octahedron:

$$\begin{pmatrix} n \\ p \\ q \\ r \end{pmatrix} = \begin{pmatrix} n-1 \\ p \\ q-1 \\ r \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q \\ r+1 \end{pmatrix} + \begin{pmatrix} n-1 \\ p-1 \\ q \\ r \end{pmatrix} + \begin{pmatrix} n-1 \\ p+1 \\ q \\ r \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q \\ r-1 \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q+1 \\ r \end{pmatrix}. \tag{11}$$

Example 1: $n = 3, p = 1, q = 2, r = 0$.

$$\begin{pmatrix} 3 \\ 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \end{pmatrix} = 3,$$

as

$$\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 2, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} = 0, \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix} = 1, \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix} = 0, \begin{pmatrix} 2 \\ 1 \\ 2 \\ -1 \end{pmatrix} = 0, \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \end{pmatrix} = 0.$$

In Figures 13 and 14 these numbers are circled in red circles, except for the number $\begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \end{pmatrix} = 0$ which goes beyond the square in Figure 13 in accordance with the expression (1); for this number $q = 3 > n = 2$.

On Figures 12 - 14 it can be seen that the octahedrons are densely filled with green cubes (branching sells). Neighboring empty cells inside octahedrons (white cubes or gaps) will be filled with green cubes at the next iteration.
Unfortunately, we cannot rotate the octahedron, as we turned the square in paragraph 3.1, since these are different geometric shapes.

4.2. Nonlinear random 3D walk in the octahedron

In paragraphs 2.2 and 3.2 we showed in details the summation of binomial coefficients for 1D and 2D cases using red arrows of different increasing length in two (1D) or four (2D) different perpendicular directions. The nonlinear binomial coefficients of the nonlinear rows (1D) or nonlinear squares (2D) $n = 1$ were obtained with the help of arrows of 1 unit length; we obtained nonlinear binomial coefficients of the series $n = 2$ using arrows of 2 unit length; we obtained nonlinear binomial coefficients of the series $n = 3$ using arrows of 3 unit length; etc.
In the proposed paper to describe a nonlinear 3D random walk we do not draw arrows (as in the monograph [14]) due to space savings. You can calculate nonlinear coefficients using the recursion formula given below and check it in the figures.
A 3D nonlinear random walk or a walk in a volume can also be described, like a linear one, using a 3D model in the form of a nonlinear arithmetic regular octahedron [19]. The octahedron has six vertices, and our nonlinear random walk has six directions. Nonlinear binomial coefficients in the computed cell of the octahedron $n = 1$ are obtained by summing the numbers of six cells adjacent to the computed cell; nonlinear binomial coefficients in the cell of the octahedron $n = 2$ we get by summing the numbers of six cells located one through the calculated cell; nonlinear binomial coefficients in the cell of the octahedron $n = 3$ we get by summing the numbers of six cells located two through the calculated cell; etc.

Figures 15 - 18 show sequentially (for the first four iterations) images of arithmetic octahedrons composed of small cubes. (a) shows images of the arithmetic octahedrons themselves, (b) shows images of layers of octahedrons composed of cubes containing numbers. These numbers correspond to the number of walks from the initial cell (initial cube) to the final cell (final cube). In Figure 4, the image of the corresponding octahedron is not represented due to space saving.

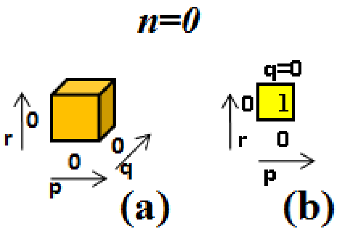

Fig. 15 The zero nonlinear arithmetic octahedron (zero iteration $n = 0$) consists of 1 cube

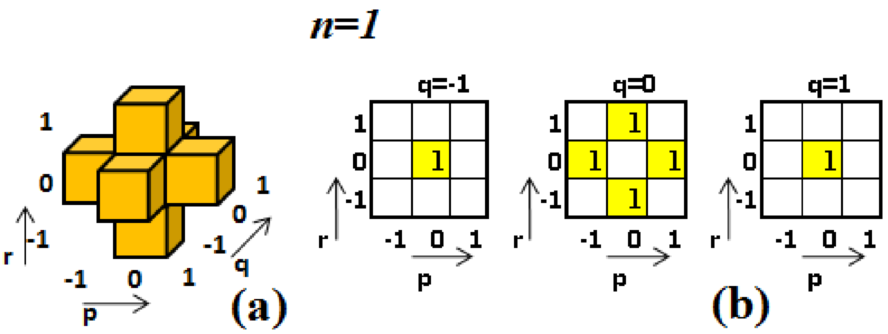

Fig. 16 The first nonlinear arithmetic octahedron (the first iteration $n = 1$) consists of 6 cubes.

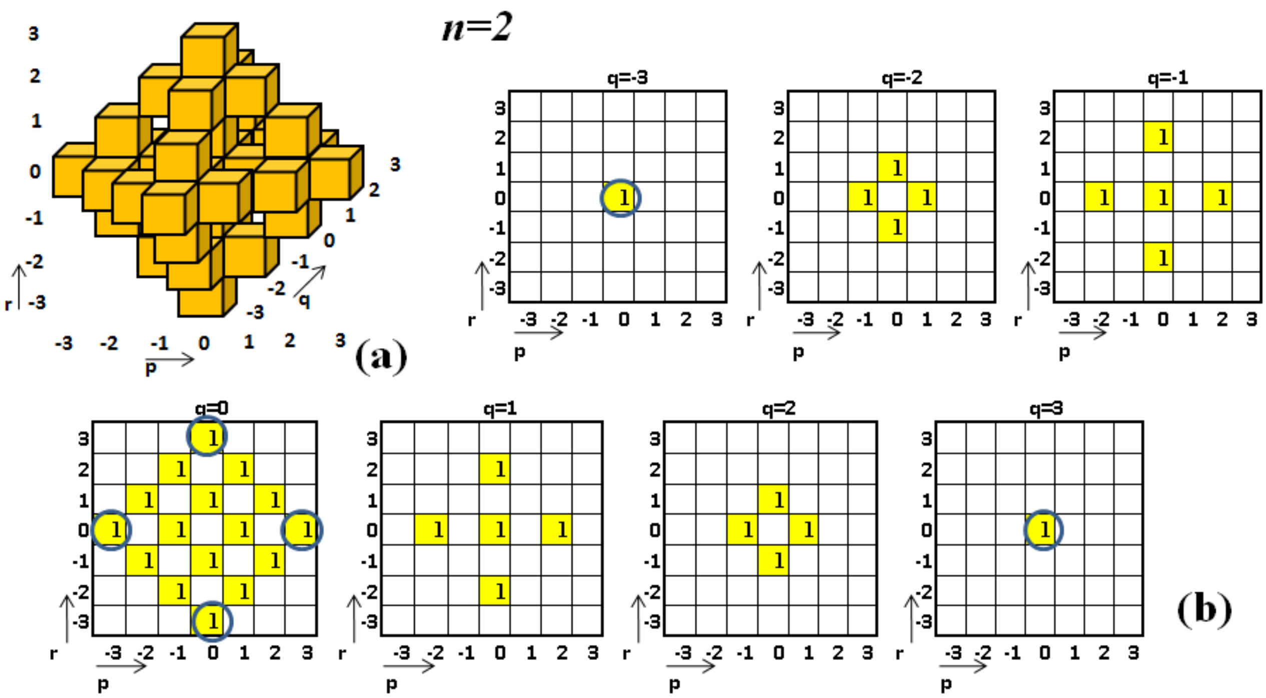

Fig. 17 The second nonlinear arithmetic octahedron (the second iteration $n = 2$) consists of 36 cubes. The Figures $q = \pm 1$ clearly show the formation of separate structures of numbers (islands of numbers).

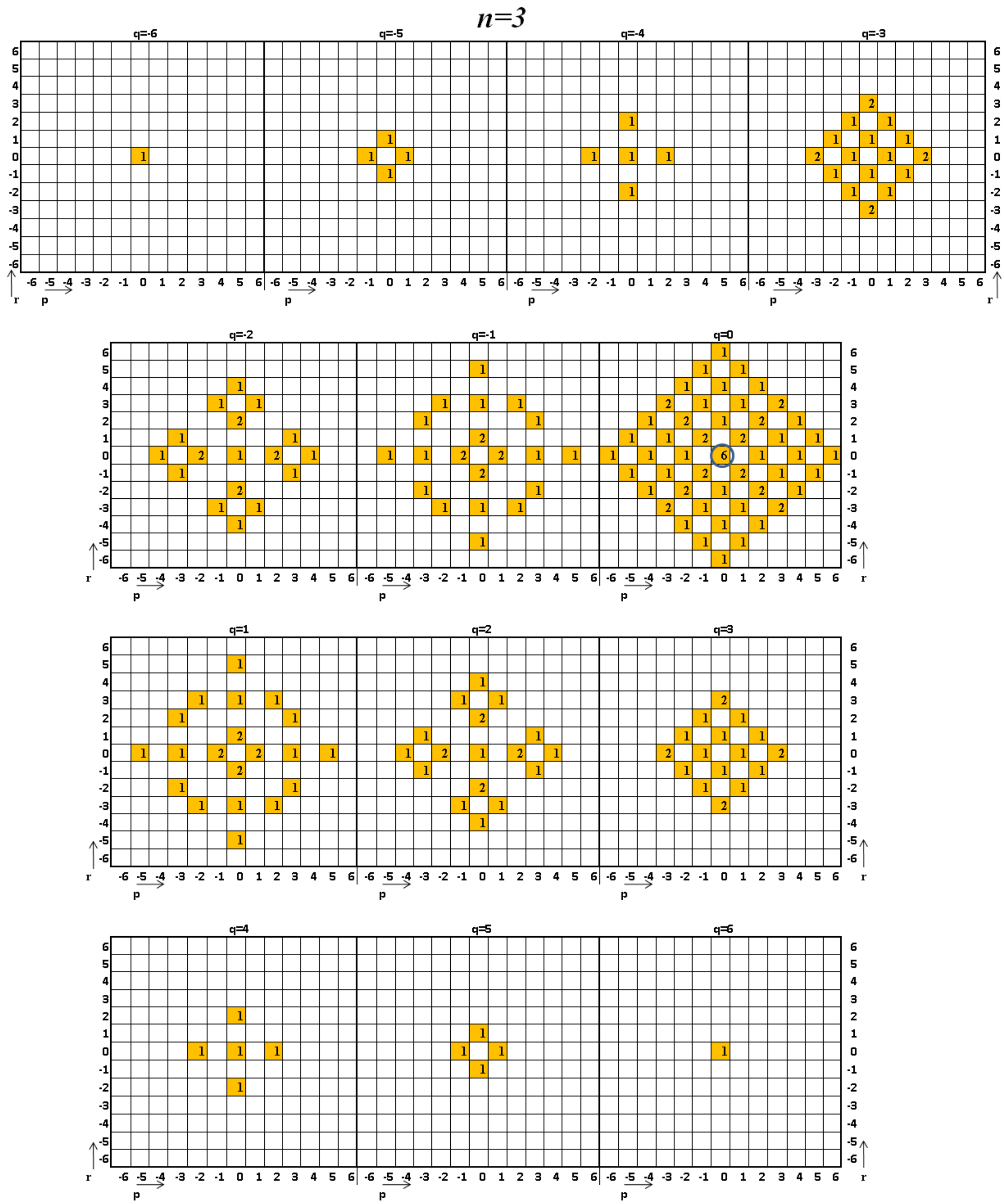

Fig. 18 The third nonlinear arithmetic octahedron (the third iteration $n = 3$) consists of 175 cubes.

The Figures $q = \pm1, q = \pm2, q = \pm4$ clearly show the formation of separate structures of numbers ("islands of numbers").

The sequence of numbers of octahedrons (rows of numbers in the octahedron) for the 3D case in this example is denoted by n: $n = 0, 1, 2, ...$ Total sum of numbers in octahedron is 6^n.

The numbers characterizing the nonlinear octahedron (the numbers characterizing the position of the cubes of which the octahedron is composed) are denoted by p, q, and r:

$$\left.\begin{array}{r} p = 0, \pm1, \pm2, \dots, \pm n(n+1)/2; \\ q = 0, \pm1, \pm2, \dots, \pm n(n+1)/2; \\ r = 0, \pm1, \pm2, \dots, \pm n(n+1)/2. \end{array}\right\} \quad (12)$$

Denote a number located in the n - octahedron as $\begin{pmatrix} n \\ p \\ q \\ r \end{pmatrix}$ then specify the number of the zero octahedron ($n = 0$) or in other words the initial conditions:

$$\begin{pmatrix} 0 \\ p \\ q \\ r \end{pmatrix} = 1 \text{ for } p = 0, q = 0, r = 0 \text{ and } \begin{pmatrix} 0 \\ p \\ q \\ r \end{pmatrix} = 0 \tag{5}$$

for the other values of p, q, and r.

We can get an expression for the consistent construction of our nonlinear arithmetic octahedron:

$$\begin{pmatrix} n \\ p \\ q \\ r \end{pmatrix} = \begin{pmatrix} n-1 \\ p \\ q-n \\ r \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q \\ r+n \end{pmatrix} + \begin{pmatrix} n-1 \\ p-n \\ q \\ r \end{pmatrix} + \begin{pmatrix} n-1 \\ p+n \\ q \\ r \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q \\ r-n \end{pmatrix} + \begin{pmatrix} n-1 \\ p \\ q+n \\ r \end{pmatrix}. \tag{6}$$

Example 2: $n = 3, p = 0, q = 0, r = 0$.

$$\begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \end{pmatrix} = 6,$$

as

$$\begin{pmatrix} 2 \\ 0 \\ -3 \\ 0 \end{pmatrix} = 1, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 3 \end{pmatrix} = 1, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 0 \end{pmatrix} = 1, \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} = 1, \begin{pmatrix} 2 \\ 0 \\ 0 \\ -3 \end{pmatrix} = 1, \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \end{pmatrix} = 1.$$

On Figures 17 and 18 these numbers are circled in blue circles.

On Figures 16 - 18 it can be seen that the octahedrons are not tightly filled with yellow cubes (branching cells). Some empty cells inside the octahedrons (white cubes or gaps) will be filled with yellow cubes at the next iteration, and some empty cells will be filled after several iterations, and some empty cells will be filled after many iterations, and so on.

Unfortunately, we cannot rotate the octahedron, as we turned the square in paragraph 3.2, as these are different geometric shapes.

5. CONCLUSIONS

Our studies of the deterministic models and visual constructions of linear (without any acceleration) and nonlinear (with the simplest uniformly acceleration) random walk and arithmetic figures given in this paper show various geometric properties and nonlinear effects of 1D, 2D, and 3D spaces.

In 1D space (Figure 2) with a linear random walk a linear arithmetic triangle (Pascal's triangle) is densely filled with numbers.

In 1D space (Figure 5) with a nonlinear random walk a nonlinear arithmetic triangle [6] is loosely (contains gaps) filled with numbers.

In 2D space (Figures 8, 10) with linear and nonlinear random walk linear and nonlinear arithmetic squares are densely filled with numbers (without gaps) in both cases.

In 3D space with a linear random walk the linear arithmetic octahedron is almost densely filled with numbers but the neighboring areas inside the octahedron remain are empty (contains gaps) until the next iteration (Figures 12 – 14).

In 3D space with a nonlinear random walk the nonlinear arithmetic octahedron is not completely filled with numbers (contains gaps) as in the case of a nonlinear 1D random walk; some neighboring regions inside the nonlinear octahedron remain empty (contains gaps) until the next iteration and some remain empty during several or many iterations. But gaps and “islands of numbers” or separate structures of numbers consistently appear and disappear after several or many iterations in a nonlinear 3D case (Figures 16 – 18).

Thus for nonlinear 1D and 3D cases we can speak of filling the numbers of the arithmetic triangle (1D) and the arithmetic octahedron (3D) in the form of “islands of numbers” or separate structures of numbers.

The results of research in this work are conveniently presented for the classification of different types of a random walk in the form of a table:

	Linear random walk (one unit steps, perpendicular each others).	**Nonlinear random walk** (first one unit steps, second two units steps, third three unit steps, ets., perpendicular each others).
1D case	• Steps of constant length along two perpendicular directions in triangle. • Gaps and “islands of numbers” are consistently absent in a linear arithmetic triangle.	• Steps of increasing length along two perpendicular directions in triangle. • Gaps and “islands of numbers” are consistently present in a nonlinear arithmetic triangle.

2D case	• Steps of constant length along four perpendicular directions in square. • Gaps and "islands of numbers" are consistently absent in a linear arithmetic square.	• Steps of increasing length along four perpendicular directions in square. • Gaps and "islands of numbers" are consistently absent in a nonlinear arithmetic square.
3D case	• Steps of constant length along six perpendicular directions in octahedron. • Gaps and "islands of numbers" are consistently appear and disappear in the neighboring areas in a linear arithmetic octahedron after each iteration.	• Steps of increasing length along six perpendicular directions in octahedron. • Gaps and "islands of numbers" are consistently appear and disappear in the different located areas in a nonlinear arithmetic octahedron after several or many iterations.

Perhaps our new geometric linear and nonlinear constructions and recursive formulas will find application to understand the development of processes in biology [8] in optics [22] and acoustics [22] and also in other areas for example in technology or medicine.

Acknowledges

The author is grateful to Arturo Tozzi, James Peters, Vasily Dikusar, Simon Shnol, Dmitry Shnol, and Alexander Ignatov for useful discussions.

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