



## New Analytical Method for Klein-Gordon Equations Arising in Quantum Field Theory

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### ABSTRACT

In this paper, the Elzaki transform homotopy perturbation method (ETHPM) has been successfully applied to obtain the approximate analytical solution of the linear and nonlinear Klein-Gordon equations which arises in quantum field theory, relativistic physics, wave theory and other physical phenomena. The proposed method is an elegant combination of the new integral transform “Elzaki transform” and homotopy perturbation method. The method is really capable of reducing the size of the computational work besides being the effective and convenient for solving linear and nonlinear partial differential equations. Some numerical examples are used to illustrate the preciseness and effectiveness of the proposed method.

**Key words:** Elzaki transform, Homotopy perturbation method, Linear and nonlinear Klein-Gordon equations.

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### 1. INTRODUCTION

In recent years, many research workers have paid attention to find the solution of nonlinear differential equations by using various methods. Nonlinear phenomena have important effects on applied mathematics, chemistry, physics and related to engineering; many such physical phenomena are modeled in terms of nonlinear partial differential equations. For example, the Klein-Gordon equations which are written in the following form

$$u_{tt}(x, t) - u_{xx}(x, t) + a u(x, t) = g(x, t), \quad (1)$$

with initial conditions

$$u(x, 0) = h(x), \quad u_t(x, 0) = f(x), \quad (2)$$

appears in quantum field theory, relativistic physics, dispersive wave-phenomena, plasma physics, nonlinear optics and applied physical sciences. Several techniques including finite difference, collocation, finite element, inverse scattering, decomposition, variational iteration method (VIM), homotopy analysis transform method (HATM) and many more, have been used to handle such equations [1, 3, 10, 11, 13, 16]. Most of these methods have their inbuilt deficiencies like the calculation of Adomain’s polynomials, the Lagrange multiplier, divergent results and huge computation work. Elzaki transform is a useful technique for solving linear partial differential equations [5] but this transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. He [9, 14, 15] developed the homotopy perturbation technique for solving such physical problems. In this paper, we use homotopy perturbation method to decompose the nonlinear term, so that the solution can be obtained by iteration procedure. This means that we can use both Elzaki transform and homotopy perturbation method to solve many nonlinear problems, see [2, 4].

### 2. STUDY OF ELZAKI TRANSFORM HOMOTOPY PERTURBATION METHOD (ETHPM)

To illustrate the basic idea of this method [2, 4]; consider a general nonlinear non-homogeneous partial differential equation with initial conditions of the form:

$$\begin{aligned} Du(x, t) + Ru(x, t) + Nu(x, t) &= g(x, t) \\ u(x, 0) &= h(x), \quad u_t(x, 0) = f(x), \end{aligned} \quad (3)$$

where  $D$  is linear differential operator of order two,  $R$  is linear differential operator of less order than  $D$ ,  $N$  is the general nonlinear differential operator and  $g(x, t)$  is the source term.

Taking Elzaki transform on both sides of eq. (3) by using [5-7], we get

$$E[Du(x, t)] + E[Ru(x, t)] + E[Nu(x, t)] = E[g(x, t)], \tag{4}$$

using the differentiation property of Elzaki transform and above initial conditions, we have

$$E[u(x, t)] = v^2 E[g(x, t)] + v^2 h(x) + v^3 f(x) - v^2 E[Ru(x, t) + Nu(x, t)] \tag{5}$$

applying the inverse Elzaki transform on both sides of eq. (5), to find

$$u(x, t) = G(x, t) - E^{-1}\{v^2 E[Ru(x, t) + Nu(x, t)]\} \tag{6}$$

where  $G(x, t)$  represents the term arising from the source term and the prescribed initial condition.

Now, we apply the homotopy perturbation method, (see [9, 14, 15])

$$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t) \tag{7}$$

and the nonlinear term can be decomposed as

$$N[u(x, t)] = \sum_{n=0}^{\infty} p^n H_n(u) \tag{8}$$

where  $H_n(u)$  are He's polynomials (see, [8, 12]) and given by

$$H_n(u_0, u_1, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} [N(\sum_{i=0}^{\infty} p^i u_i)]_{p=0}, \quad n = 0, 1, 2, \dots \tag{9}$$

substituting eqs. (7) and (8) in eq. (6), we get

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = G(x, t) - p\{E^{-1}[v^2 E(R \sum_{n=0}^{\infty} p^n u_n(x, t) + \sum_{n=0}^{\infty} p^n H_n(u))]\}. \tag{10}$$

This is the coupling of the Elzaki transform and the homotopy perturbation method. Comparing the coefficient of like powers of  $p$ , the following approximations are obtained.

$$\begin{aligned} p^0: u_0(x, t) &= G(x, t) \\ p^1: u_1(x, t) &= -E^{-1}\{v^2 E[Ru_0(x, t) + H_0(u)]\} \\ p^2: u_2(x, t) &= -E^{-1}\{v^2 E[Ru_1(x, t) + H_1(u)]\} \\ p^3: u_3(x, t) &= -E^{-1}\{v^2 E[Ru_2(x, t) + H_2(u)]\} \\ &\dots \end{aligned}$$

Then the solution is

$$\begin{aligned} u(x, t) &= \log_{p \rightarrow 1} \sum_{n=0}^{\infty} u_n(x, t) \\ &= u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \dots \end{aligned} \tag{11}$$

### 3. APPLICATIONS

In this section, we apply the Elzaki transform homotopy perturbation method (ETHPM) to solve the linear and nonlinear Klein-Gordon equations.

**Example 3.1.** Consider the following linear Klein-Gordon equation

$$u_{tt}(x, t) - u_{xx}(x, t) + u(x, t) = 0, \tag{12}$$

with the initial conditions

$$u(x, 0) = 0, \quad u_t(x, 0) = x. \tag{13}$$

Applying the Elzaki transform on both sides of eq. (12) subject to the initial conditions (13), we have

$$E[u(x, t)] = v^3 x + v^2 E\left[\frac{\partial^2 u}{\partial x^2} - u\right]. \tag{14}$$

The inverse of Elzaki transform implies that

$$u(x, t) = xt + E^{-1}\left\{v^2 E\left[\frac{\partial^2 u}{\partial x^2} - u\right]\right\}. \tag{15}$$

Now we apply the homotopy perturbation method,

$$u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t), \tag{16}$$

and the nonlinear term can be decomposed as

$$N[u(x, t)] = \sum_{n=0}^{\infty} p^n H_n(u), \tag{17}$$

using eqs. (16)-(17) into eq. (15), we get

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = xt + pE^{-1}\{v^2 E[\sum_{n=0}^{\infty} p^n u_n(x, t)]\}. \tag{18}$$

Comparing the coefficients of like powers of  $p$  in (18), we have

$$\begin{aligned} p^0: u_0(x, t) &= xt, \\ p^1: u_1(x, t) &= E^{-1}\left\{v^2 E\left[\frac{\partial^2 u_0}{\partial x^2} - u_0\right]\right\} = -\frac{xt^3}{3!}, \\ p^2: u_2(x, t) &= E^{-1}\left\{v^2 E\left[\frac{\partial^2 u_1}{\partial x^2} - u_1\right]\right\} = \frac{xt^5}{5!}, \end{aligned}$$

proceeding in similar manner we can obtain further values,

$$\begin{aligned} p^3: u_3(x, t) &= -\frac{xt^7}{7!}, \\ p^4: u_4(x, t) &= \frac{xt^9}{9!}, \\ &\dots \end{aligned}$$

Therefore the solution  $u(x, t)$  is given by

$$u(x, t) = x \left( t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \dots \right),$$

$$\Rightarrow u(x, t) = x \sin t, \tag{19}$$

which is the same solution as obtained by VIM [13] and HATM [10].

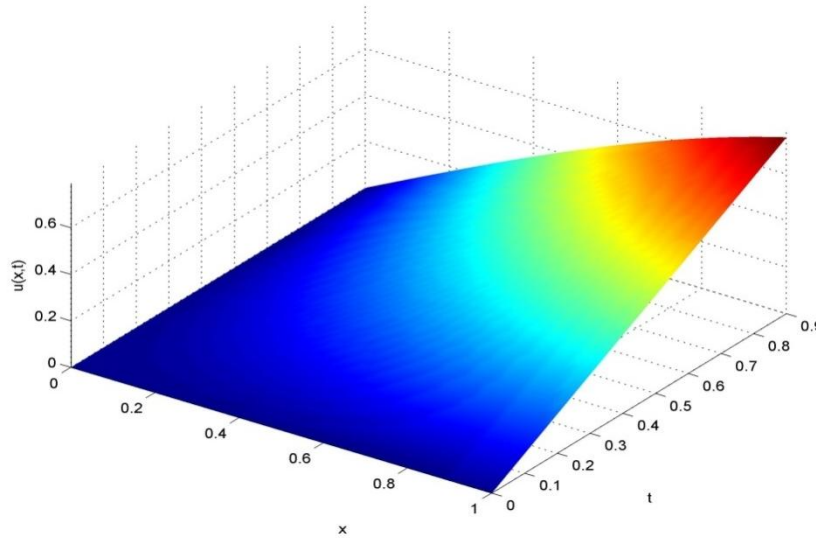


Fig. 1 Graph of  $u(x, t) = x \sin t$ ,  $t > 0$  and  $0 \leq x \leq 1$ .

**Example 3.2.** Consider the following linear Klein-Gordon equation

$$u_{tt}(x, t) - u_{xx}(x, t) + u(x, t) = 2 \sin x, \tag{20}$$

with the initial conditions

$$u(x, 0) = \sin x, \quad u_t(x, 0) = 1. \tag{21}$$

Applying the Elzaki transform on both sides of eq. (20) subject to the initial conditions (21), we have

$$E[u(x, t)] = v^2 \sin x + v^3 + 2v^4 \sin x + v^2 E \left[ \frac{\partial^2 u}{\partial x^2} - u \right]. \tag{22}$$

The inverse of Elzaki transform implies that

$$u(x, t) = \sin x + t + t^2 \sin x + E^{-1} \left\{ v^2 E \left[ \frac{\partial^2 u}{\partial x^2} - u \right] \right\}. \tag{23}$$

Using eqs. (16)-(17) into eq. (23), we get

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = \sin x + t + t^2 \sin x + p E^{-1} \{ v^2 E [\sum_{n=0}^{\infty} p^n u_n(x, t)] \}. \tag{24}$$

Comparing the coefficients of like powers of  $p$  in (24), we get

$$p^0: u_0(x, t) = \sin x + t + t^2 \sin x,$$

$$p^1: u_1(x, t) = E^{-1} \left\{ v^2 E \left[ \frac{\partial^2 u_0}{\partial x^2} - u_0 \right] \right\}$$

$$= -t^2 \sin x - \frac{t^3}{3!} - \frac{t^4}{3!} \sin x,$$

$$p^2: u_2(x, t) = E^{-1} \left\{ v^2 E \left[ \frac{\partial^2 u_1}{\partial x^2} - u_1 \right] \right\}$$

$$= \frac{t^4}{3!} \sin x + 8 \frac{t^6}{6!} \sin x + \frac{t^5}{5!},$$

$$p^3: u_3(x, t) = E^{-1} \left\{ v^2 E \left[ \frac{\partial^2 u_2}{\partial x^2} - u_2 \right] \right\}$$

$$= -8 \frac{t^6}{6!} \sin x - 16 \frac{t^8}{8!} \sin x - \frac{t^7}{7!},$$

proceeding in similar manner we can obtain further values,

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Therefore the solution  $u(x, t)$  is given by

$$\Rightarrow u(x, t) = \sin x + \sin t, \tag{25}$$

which is the same solution as obtained by VIM [13] and HATM [10].

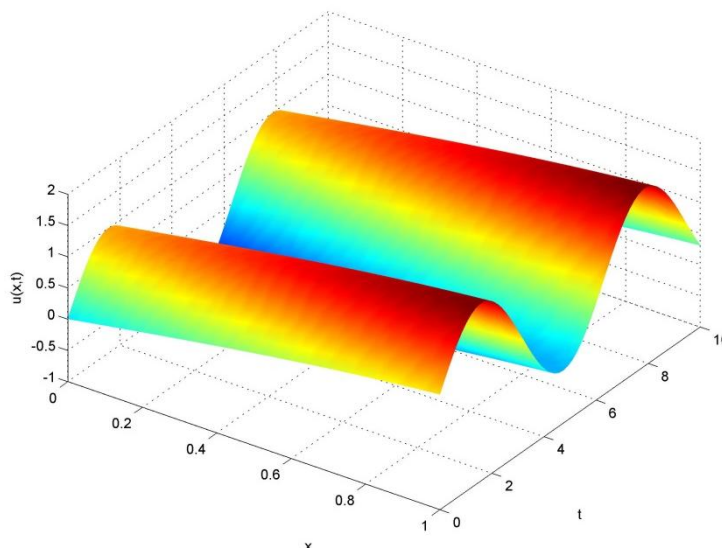


Fig. 2 Graph of  $u(x,t) = \sin x + \sin t$ ,  $t > 0$  and  $0 \leq x \leq 1$ .

**Example 3.3.** Consider the following nonlinear Klein-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) + u^2(x,t) = x^2 t^2, \tag{26}$$

with the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = x. \tag{27}$$

Applying the Elzaki transform on both sides of eq. (26) subject to the initial conditions (27), we have

$$E[u(x,t)] = xv^3 + 2x^2v^6 + v^2E\left[\frac{\partial^2 u}{\partial x^2} - u^2\right]. \tag{28}$$

The inverse of Elzaki transform implies that

$$u(x,t) = xt + 2x^2 \frac{t^4}{4!} + E^{-1}\left\{v^2E\left[\frac{\partial^2 u}{\partial x^2} - u^2\right]\right\}. \tag{29}$$

Now we apply the homotopy perturbation method,

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t), \tag{30}$$

and the nonlinear term can be decomposed as

$$N[u(x,t)] = \sum_{n=0}^{\infty} p^n H_n(u), \tag{31}$$

using eqs. (30)-(31) into eq. (29), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = xt + 2x^2 \frac{t^4}{4!} + pE^{-1}\left\{v^2E\left[\sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right\}. \tag{32}$$

Where  $H_n(u)$  are He's polynomials [8, 12] that represents the nonlinear terms. The first few components of He's polynomials, are given by

$$\begin{aligned} H_0(u) &= (u_0)^2, \\ H_1(u) &= 2u_0u_1, \\ H_2(u) &= 2u_0u_1 + u_1^2, \end{aligned}$$

...

Comparing the coefficients of like powers of  $p$  in (32), we get

$$\begin{aligned} p^0: u_0(x,t) &= xt + \frac{x^2 t^4}{12}, \\ p^1: u_1(x,t) &= E^{-1}\left\{v^2E\left[\frac{\partial^2 u_0}{\partial x^2} - H_0(u)\right]\right\} \\ &= \frac{t^6}{180} - \frac{x^4 t^{10}}{12960} - \frac{x^3 t^7}{252} - \frac{x^2 t^4}{12}, \\ p^2: u_2(x,t) &= E^{-1}\left\{v^2E\left[\frac{\partial^2 u_1}{\partial x^2} - H_1(u)\right]\right\} \\ &= -\frac{x^2 t^{12}}{71280} - \frac{11xt^9}{22680} - \frac{t^6}{180} + \frac{x^6 t^{16}}{18662400} + \frac{11x^4 t^{10}}{45360} + \frac{383x^5 t^{13}}{15921360} + \frac{x^3 t^7}{252}, \end{aligned}$$

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Proceeding in similar manner we can obtain further values. Therefore the solution  $u(x,t)$  is given by

$$\Rightarrow u(x,t) = xt, \tag{33}$$

which is the same solution as obtained by VIM [13] and HATM [10].

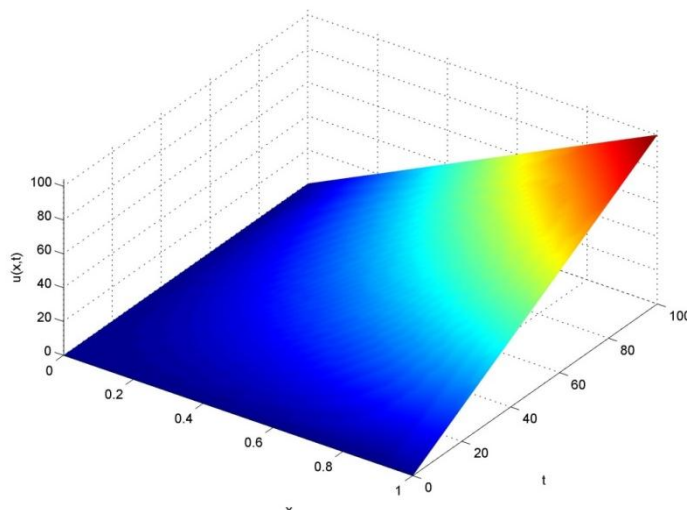


Fig. 3 Graph of  $u(x, t) = xt$ ,  $t > 0$  and  $0 \leq x \leq 1$ .

**Example 3.4.** Consider the following nonlinear Klein-Gordon equation

$$u_{tt}(x, t) - u_{xx}(x, t) + u^2(x, t) = 2x^2 - 2t^2 + x^4t^4, \tag{34}$$

with the initial conditions

$$u(x, 0) = 0, \quad u_t(x, 0) = 0. \tag{35}$$

Applying the Elzaki transform on both sides of eq. (34) subject to the initial conditions (35), we have

$$E[u(x, t)] = 2x^2v^4 - 4v^6 + 24x^4v^8 + v^2E\left[\frac{\partial^2 u}{\partial x^2} - u^2\right]. \tag{36}$$

The inverse of Elzaki transform implies that

$$u(x, t) = x^2t^2 - \frac{t^4}{6} + \frac{x^4t^6}{30} + E^{-1}\left\{v^2E\left[\frac{\partial^2 u}{\partial x^2} - u^2\right]\right\}. \tag{37}$$

Using eqs. (30)-(31) into eq. (37), we get

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = x^2t^2 - \frac{t^4}{6} + \frac{x^4t^6}{30} + pE^{-1}\{v^2E[\sum_{n=0}^{\infty} p^n u_n(x, t) - \sum_{n=0}^{\infty} p^n H_n(u)]\}, \tag{38}$$

proceeding in similar manner as we done in solution of Example 3.3. Again comparing the coefficients of like power of  $p$  in eq. (38), we have

$$p^0: u_0(x, t) = x^2t^2 - \frac{t^4}{6} + \frac{x^4t^6}{30},$$

$$p^1: u_1(x, t) = E^{-1}\left\{v^2E\left[\frac{\partial^2 u_0}{\partial x^2} - H_0(u)\right]\right\} \\ = \frac{t^4}{6} - \frac{x^4t^6}{30} - \frac{288x^2t^8}{40320} - \frac{20t^8}{40320} - \frac{532224x^8t^{14}}{14!} + \frac{24x^2t^7}{5040} - \frac{2688x^6t^{10}}{3628800} + \frac{4032x^4t^{11}}{39916800},$$

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Proceeding in similar manner we can obtain further values. Therefore the solution  $u(x, t)$  is given by

$$u(x, t) = x^2t^2, \tag{39}$$

which is the same solution as obtained by VIM [13] and HATM [10].

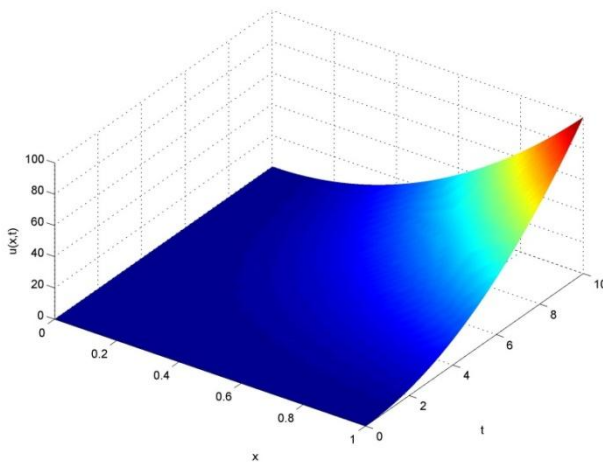


Fig. 4 Graph of  $u(x, t) = x^2t^2$ ,  $t > 0$  and  $0 \leq x \leq 1$ .

**Example 3.5.** Consider the following nonlinear Klein-Gordon equation

$$u_{tt}(x, t) - u_{xx}(x, t) + u^2(x, t) = 6xt(x^2 - t^2) + x^6t^6, \quad (40)$$

with the initial conditions

$$u(x, 0) = 0, \quad u_t(x, 0) = 0. \quad (41)$$

Applying the Elzaki transform on both sides of eq. (40) subject to the initial conditions (41), we have

$$E[u(x, t)] = 6x^3v^5 - 36xv^7 + 720x^6v^{10} + v^2E\left[\frac{\partial^2u}{\partial x^2} - u^2\right]. \quad (42)$$

The inverse of Elzaki transform implies that

$$u(x, t) = x^3t^3 - \frac{3xt^5}{10} + \frac{x^6t^8}{56} + E^{-1}\left\{v^2E\left[\frac{\partial^2u}{\partial x^2} - u^2\right]\right\}. \quad (43)$$

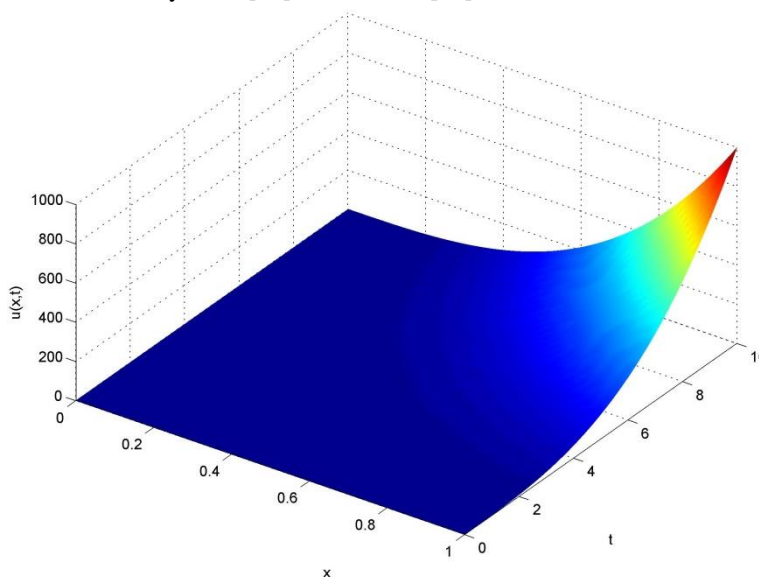
Using eqs. (30)-(31) into eq. (43), we get

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = x^3t^3 - \frac{3xt^5}{10} + \frac{x^6t^8}{56} + pE^{-1}\{v^2E[\sum_{n=0}^{\infty} p^n u_n(x, t) - \sum_{n=0}^{\infty} p^n H_n(u)]\}. \quad (44)$$

In a similar manner as before, the solution  $u(x, t)$  is given by

$$u(x, t) = x^3t^3, \quad (45)$$

which is the same solution as obtained by VIM [13] and HATM [10].



**Fig. 5** Graph of  $u(x, t) = x^3t^3$ ,  $t > 0$  and  $0 \leq x \leq 1$ .

#### 4. CONCLUSION

In this paper, the mixture of new integral transform “Elzaki transform” with the homotopy perturbation method has been successfully applied to find the solution of the linear and nonlinear Klein-Gordon equations with initial conditions. The method is reliable and easy to use. The results show that the Elzaki transform homotopy perturbation method (ETHPM) is powerful and efficient technique in finding exact and approximate solutions for linear and nonlinear partial differential equations. In conclusion, the ETHPM may be considered as a nice refinement in existing numerical techniques and might find the wide applications.

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