Cubic Transmuted Gompertz-Makeham Distribution

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ABSTRACT

We introduce a cubic rank transmuted Gompertz-Makeham distribution that extends the standard Gompertz-Makeham model by incorporating two more parameters into its distribution functions. We study the main statistical properties of the cubic transmuted model, including its hazard rate function, moment-generating function, moments, characteristic function, quantile function, entropy, order statistics, and moments of order statistics.

Key words: Gompertz-Makeham distribution, mixture of distribution, entropy, order statistics, moments of order statistics
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1. INTRODUCTION

In 1860, Makeham [1] introduced the Gompertz-Makeham probability distribution as an extension of the Gompertz probability distribution that was introduced by Gompertz [2] in 1825. The Gompertz-Makeham distribution is a continuous probability distribution that has widely been used in survival analysis and modeling human mortality. It has been recently used together with its extensions in many fields of sciences. See Marshall and Olkin [3] for a comprehensive review of the history and theory of the Gompertz-Makeham probability distribution. Gompertz [5] emphasizes the practical importance of this probability distribution. Detailed information about the Gompertz-Makeham distribution, its mathematical and statistical properties, and its applications can be found in Johnson et al. [5] and [6].

A random variable X is said to have a Gompertz-Makeham distribution with strictly positive real parameters α, β and γ, abbreviated as $X : GM(\alpha, \beta, \gamma)$, if its cumulative distribution function (cdf) is given by

$$F_{GM}(x; \alpha, \beta, \gamma) = 1 - \exp\left\{-\gamma x - (\alpha/\beta)\left(e^{\beta x} - 1\right)\right\}, \quad x > 0. \qquad (1.1)$$

The corresponding probability density function (pdf) is given as

$$f_{GM}(x; \alpha, \beta, \gamma) = (\gamma + \alpha e^{\beta x}) \exp\left\{(\alpha/\beta)\left(1 - e^{\beta x}\right) - \gamma x\right\}, \quad x > 0. \qquad (1.2)$$

Given a baseline distribution with cdf $G(x)$, the transmutation map of cubic rank (CT) distribution has a cdf given by

$$F(x) = 1 - [1 - G(x)](1 - \lambda_1 + (\lambda_1 - \lambda_2)[1 - G(x)] + \lambda_2[1 - G(x)]^2), \qquad (1.3)$$

where $0 \le \lambda_1 \le 1$ and $\lambda_1 - 1 \le \lambda_2 \le \lambda_1$.

The corresponding pdf is given by

$$f(x) = g(x)[1 - G(x)](1 + \lambda_1[1 - 2G(x)] + \lambda_2[1 - 4G(x)] + 3\lambda_2 G(x)^2), \qquad (1.4)$$

where $g(x)$ is the pdf of the baseline distribution.

Recently, many authors have extend the GM model by transmutation. In 2017, khan et al. [7] introduced the four parameter transmuted generalized Gompertz distribution with a cdf and pdf given respectively by

$$F(x; \alpha, \beta, \eta) = [1 - \exp\{-\frac{\alpha}{\eta}(e^{\eta x} - 1)\}]^\beta \{1 + \lambda - \lambda[1 - \exp\{-\frac{\alpha}{\eta}(e^{\eta x} - 1)\}]^\beta\},$$

$$f(x; \alpha, \beta, \eta) = \alpha\beta e^{\eta x} \exp\{-\frac{\alpha}{\eta}(e^{\eta x} - 1)\}[1 - \exp\{-\frac{\alpha}{\eta}(e^{\eta x} - 1)\}]^{\beta-1}$$

$$\times\{1 + \lambda - \lambda[1 - \exp\{-\frac{\alpha}{\eta}(e^{\eta x} - 1)\}]^\beta\},$$

where $\alpha, \beta, \eta > 0, \lambda \le 1$ and $x > 0$.

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Abdul-Moniem and Seham [8] introduced the transmuted Gompertz (TGD) distribution and studied its statistical properties. The cumulative density function (cdf) of (TGD) distribution is

$$F_{TGD}(x;\alpha,\theta) = [1 - e^{-\theta(e^{\alpha x}-1)}][1 + \lambda e^{-\theta(e^{\alpha x}-1)}], \qquad (1.5)$$

where $x \geq 0$, $\alpha > 0, \theta > 0$ and $|\lambda| \leq 1$.

The corresponding probability density function (pdf) of (TGD) distribution is

$$f_{TGD}(x;\alpha,\theta) = \alpha\theta e^{\alpha x} e^{-\theta(e^{\alpha x}-1)}[1 - \lambda + 2\lambda e^{-\theta(e^{\alpha x}-1)}].$$

El-Bar [9] introduced an extended Gompertz-Makeham model and studied its properties. It is in fact a transmuted Gompertz-Makeham (TGM) distribution that has a cdf

$$F_{TGM}(x;\alpha,\beta,\gamma) = \{1 - e^{(\alpha/\beta)(1-e^{\beta x})-\gamma x}\}\{1 + \lambda e^{(\alpha/\beta)(1-e^{\beta x})-\gamma x}\}, \qquad (1.6)$$

where $\alpha > 0, \beta > 0, \gamma > 0$, and $|\lambda| \leq 1$.

The corresponding probability density function pdf is given by

$$f_{TGM}(x;\alpha,\beta,\gamma) = (\gamma + \alpha e^{\beta x})e^{(\alpha/\beta)(1-e^{\beta x})-\gamma x}\{1 - \lambda + 2\lambda e^{(\alpha/\beta)(1-e^{\beta x})-\gamma x}\}, \ x > 0.$$

Riffi [10] generalized the TGM distribution to a generalized model of the transmuted Gompertz-Makeham distributions (GTGM) by adding two more parameters to its distribution functions, and studied some of the properties of GTGM distribution. The cdf of the GTGM distribution is given by

$$F_X(x;\Omega) = 1 - e^{(\alpha/\beta)\delta(1-e^{\beta x})-\gamma\delta x}\{1 - \lambda + \lambda e^{(\alpha/\beta)\varepsilon(1-e^{\beta x})-\gamma\varepsilon x}\},$$

where Ω is the vector $(\alpha,\beta,\gamma,\delta,\varepsilon,\lambda), \delta \geq 1, \varepsilon > 0, \alpha > 0, \beta > 0, \gamma > 0$, and $|\lambda| \leq 1$.

The corresponding probability density function pdf, for $x > 0$ is

$$f_X(x;\Omega) = (\gamma + \alpha e^{\beta x})e^{(\delta/\beta)\alpha[(1-e^{\beta x})-\alpha\gamma x]}\{\delta(1-\lambda) + \lambda(\delta+\varepsilon)e^{(\alpha/\beta)\varepsilon(1-e^{\beta x})-\gamma\varepsilon x}\}.$$

In this paper, we will introduce the cubic rank transmuted Gompertz-Makeham (CTGM) distribution by substituting $G(x)$ in eq.n(1.3) to get the cdf of (CTGM) distribution

$$F(x;\Theta) = [1 - e^{\frac{\alpha}{\beta}(1-e^{\beta x})-\gamma x}][1 + \lambda_1 e^{\frac{\alpha}{\beta}(1-e^{\beta x})-\gamma x} + \lambda_2 e^{\frac{2\alpha}{\beta}(1-e^{\beta x})-2\gamma x}], \qquad (1.7)$$

where Θ is the vector $(\alpha,\beta,\gamma,\lambda_1,\lambda_2)$, $\alpha > 0, \beta > 0, \gamma > 0$, $0 \leq \lambda_1 \leq 1$ and
$\lambda_1 - 1 \leq \lambda_2 \leq \lambda_1$,

and the corresponding pdf

$$f(x;\Theta) = (\gamma + \alpha e^{\beta x})e^{\frac{\alpha}{\beta}(1-e^{\beta x})-\gamma x}[1 - \lambda_1 + 2(\lambda_1 - \lambda_2)e^{\frac{\alpha}{\beta}(1-e^{\beta x})-\gamma x} + 3\lambda_2 e^{\frac{2\alpha}{\beta}(1-e^{\beta x})-2\gamma x}]. \qquad (1.8)$$

Plot of the CTGM pdf (left) and corresponding cdf (right) for a variety of values of its parameters.

2. SUB-MODELS AND POSSIBLE EXTENSION

1. If we let $\lambda_1 = \lambda_2 = 0$ in (1.7), then we get the standard (GM) distribution with parameters α, β, γ.

2. If we let $\lambda_2 = \gamma = 0$ and $\theta = \dfrac{\alpha}{\beta}$ in (1.7), then we get the (TGD) distribution described in (1.5).

3. If we let $\lambda_2 = 0$ in (1.7), then we get the (TGM) distribution described in (1.6).

From the higher rank transmuted (HRT-G) family of distributions which proposed by Riffi [11], we can generalize the cubic rank transmuted Gompertz-Makeham (CTGM) distribution to the generalized cubic rank transmuted Gompertz-Makeham (GCTGM) distribution with cdf

$$F(x) = 1 - [1 - G(x)]^{\alpha_1}\{1 - \lambda_1 + (\lambda_1 - \lambda_2)[1 - G(x)]^{\alpha_2} + \lambda_2[1 - G(x)]^{\alpha_3}\} \qquad (2.1)$$

where $\alpha_1 \geq 1, \alpha_2, \alpha_3 \geq 0$, and $\lambda_1 - 1 \leq \lambda_2 \leq \lambda_1$.

It is possible to extend the model by exponentiated the GM distribution that we use as a baseline; i.e., we replace $G(x)$ by $G(x)^a$ in (2.1), where $a > 0$. That is, the cdf of the extended model will be

$$F_E(x) = 1 - [1 - G(x)^a]^{\alpha_1}\{1 - \lambda_1 + (\lambda_1 - \lambda_2)[1 - G(x)^a]^{\alpha_2} + \lambda_2[1 - G(x)^a]^{\alpha_3}\}. \qquad (2.2)$$

3. GTGM AS MIXTURE OF DISTRIBUTION

From (1.8) we cane write the pdf of CTGM distribution as

$$f_X(x;\Omega) = (1-\lambda_1)(\gamma + \alpha\, e^{\beta x})e^{\frac{\alpha}{\beta}(1-e^{\beta x})-\gamma x}$$

$$+ (\lambda_1 - \lambda_2)(2\gamma + 2\alpha\, e^{\beta x})e^{\frac{2\alpha}{\beta}(1-e^{\beta x})-2\gamma x}$$

$$+ \lambda_2(3\gamma + 3\alpha\, e^{\beta x})e^{\frac{3\alpha}{\beta}(1-e^{\beta x})-3\gamma x}.$$

Hence the CTGM distribution is a mixture of three GM distributions with weights $(1-\lambda_1)$, $(\lambda_1 - \lambda_2)$ and λ_2 as in the described in the following equation

$$f_X(x;\Omega) = (1-\lambda_1)f_1(x;\Omega) + (\lambda_1 - \lambda_2)f_2(x;\Omega) + \lambda_2 f_3(x;\Omega),$$

where the functions $f_1(x;\Omega)$, $f_2(x;\Omega)$ and $f_3(x;\Omega)$ are given by

$$f_i(x;\Omega) = (i\gamma + i\alpha\, e^{\beta x})e^{\frac{i\alpha}{\beta}(1-e^{\beta x})-i\gamma x}, \quad for\ i = 1,2,3.$$

Here, $f_i(x;\Omega)$ is the pdf of a GM random variable with parameters $i\alpha, \beta\ and\ i\gamma$ for $i = 1,2,3$.

4. HAZARD RATE FUNCTION

The survivor function of the cdf $F(x)$ of distribution is defined by $S(x) = 1 - F(x)$.

For the cubic rank transmuted Gompertz-Makeham probability distribution, the survivor function is given as,

$$S(x) = 1 - [1 - e^{\frac{\alpha}{\beta}(1-e^{\beta x})-\gamma x}][1 + \lambda_1\, e^{\frac{\alpha}{\beta}(1-e^{\beta x})-\gamma x} + \lambda_2\, e^{\frac{2\alpha}{\beta}(1-e^{\beta x})-2\gamma x}].$$

The hazard rate function can be written as the ratio between the pdf $f(x)$ and the survivor function $S(x) = 1 - F(x)$. That is,

$$h(x) = \frac{f(x)}{S(x)},$$

then we can fined the hazard rate function of CTGM distribution by (1.7) and (1.8):

$$h(x) =$$

$$\frac{(\gamma + \alpha\, e^{\beta x})e^{\frac{\alpha}{\beta}(1-e^{\beta x})-\gamma x}[1 - \lambda_1 + 2(\lambda_1 - \lambda_2)e^{\frac{\alpha}{\beta}(1-e^{\beta x})-\gamma x} + 3\lambda_2\, e^{\frac{2\alpha}{\beta}(1-e^{\beta x})-2\gamma x}]}{1 - [1 - e^{\frac{\alpha}{\beta}(1-e^{\beta x})-\gamma x}][1 + \lambda_1\, e^{\frac{\alpha}{\beta}(1-e^{\beta x})-\gamma x} + \lambda_2\, e^{\frac{2\alpha}{\beta}(1-e^{\beta x})-2\gamma x}]}.$$

Plot of the CTGM hazard function for a variety of values of its parameters.

5. MOMENTS AND MOMENT-GENERATING FUNCTION

The moment-generating function (mgf) of the CTGM distribution can be calculated using the transformation of variables technique. The result will be given in terms of the generalized integro-exponential function which is defined by

$$E_{(s)}^r(z) = \frac{1}{\Gamma(r+1)}\int_1^\infty \log^r(u)u^{-s}e^{-zu}\,du, z > 0. \tag{5.1}$$

Below, we are going to use the following identity from [?]

$$E_{(s)}^{r-1}(z) = (s-1)E_{(s)}^r(z) + zE_{(s-1)}^r(z), z > 0. \tag{5.2}$$

Theorem 5.1 *Let* $X : CTGM(\Theta)$. *Then the moment-generating function (mgf) of* X, *is given by*

$$M_X(t) = \frac{\gamma(1-\lambda_1)}{\beta}e^{\frac{\alpha}{\beta}}E_{(\frac{\gamma-t}{\beta}+1)}^0\left(\frac{\alpha}{\beta}\right) + \frac{2\gamma(\lambda_1 - \lambda_2)}{\beta}e^{\frac{2\alpha}{\beta}}E_{(\frac{2\gamma-t}{\beta}+1)}^0\left(\frac{2\alpha}{\beta}\right)$$

$$+ \frac{3\gamma\lambda_2}{\beta}e^{\frac{3\alpha}{\beta}}E_{(\frac{3\gamma-t}{\beta}+1)}^0\left(\frac{3\alpha}{\beta}\right) + \frac{\alpha(1-\lambda_1)}{\beta}e^{\frac{\alpha}{\beta}}E_{(\frac{\gamma-t}{\beta})}^0\left(\frac{\alpha}{\beta}\right)$$

$$+\frac{2\alpha(\lambda_1-\lambda_2)}{\beta}e^{\frac{\alpha}{\beta}}E^0_{(\frac{2\gamma-t}{\beta})}(\frac{2\alpha}{\beta})+\frac{3\alpha\lambda_2}{\beta}e^{\frac{3\alpha}{\beta}}E^0_{(\frac{3\gamma-t}{\beta})}(\frac{3\alpha}{\beta}).\tag{5.3}$$

Proof.

$$M_X(t)=E(e^{tX})=\int_{-\infty}^{\infty}e^{tx}f(x;\Theta)\,dx,\quad then\ by\ (1.8)$$

$$=\int_{-\infty}^{\infty}e^{tx}\,(\gamma+\alpha e^{\beta x})\{(1-\lambda_1)e^{[\frac{\alpha}{\beta}(1-e^{\beta x})-\gamma x]}$$

$$+2(\lambda_1-\lambda_2)e^{[\frac{2\alpha}{\beta}(1-e^{\beta x})-2\gamma x]}+3\lambda_2\,e^{[\frac{3\alpha}{\beta}(1-e^{\beta x})-3\gamma x]}\}\,dx.$$

Then, in terms of the generalized integro-exponential function, $E^r_{(s)}(z)$, the mgf of X can be written as

$$M_X(t)=\frac{\gamma(1-\lambda_1)}{\beta}e^{\frac{\alpha}{\beta}}E^0_{(\frac{\gamma-t}{\beta}+1)}(\frac{\alpha}{\beta})+\frac{2\gamma(\lambda_1-\lambda_2)}{\beta}e^{\frac{2\alpha}{\beta}}E^0_{(\frac{2\gamma-t}{\beta}+1)}(\frac{2\alpha}{\beta})$$

$$+\frac{3\gamma\lambda_2}{\beta}e^{\frac{3\alpha}{\beta}}E^0_{(\frac{3\gamma-t}{\beta}+1)}(\frac{3\alpha}{\beta})+\frac{\alpha(1-\lambda_1)}{\beta}e^{\frac{\alpha}{\beta}}E^0_{(\frac{\gamma-t}{\beta})}(\frac{\alpha}{\beta})$$

$$+\frac{2\alpha(\lambda_1-\lambda_2)}{\beta}e^{\frac{\alpha}{\beta}}E^0_{(\frac{2\gamma-t}{\beta})}(\frac{2\alpha}{\beta})+\frac{3\alpha\lambda_2}{\beta}e^{\frac{3\alpha}{\beta}}E^0_{(\frac{3\gamma-t}{\beta})}(\frac{3\alpha}{\beta}).$$

corollary 5.1 *The k th partial derivative $M_X(t)$ with respect to t is*

$$M_X^{(k)}(t)=\beta^{-(k+1)}\,k!\{\gamma(1-\lambda_1)\,e^{\frac{\alpha}{\beta}}\,E^k_{(\frac{\gamma-t}{\beta}+1)}(\frac{\alpha}{\beta})+2\gamma(\lambda_1-\lambda_2)\,e^{\frac{2\alpha}{\beta}}\,E^k_{(\frac{2\gamma-t}{\beta}+1)}(\frac{2\alpha}{\beta})$$

$$+3\gamma\lambda_2\,e^{\frac{3\alpha}{\beta}}\,E^k_{(\frac{3\gamma-t}{\beta}+1)}(\frac{3\alpha}{\beta})+\alpha(1-\lambda_1)\,e^{\frac{\alpha}{\beta}}\,E^k_{(\frac{\gamma-t}{\beta})}(\frac{\alpha}{\beta})$$

$$+2\alpha(\lambda_1-\lambda_2)\,e^{\frac{2\alpha}{\beta}}\,E^k_{(\frac{2\gamma-t}{\beta})}(\frac{2\alpha}{\beta})+3\alpha\lambda_2\,e^{\frac{3\alpha}{\beta}}\,E^k_{(\frac{3\gamma-t}{\beta})}(\frac{3\alpha}{\beta})\}.\tag{5.4}$$

Proof. Using the equation

$$\frac{\partial^k}{\partial t^k}E^0_{(\frac{i\gamma-t}{\beta}+j)}(\frac{i\alpha}{\beta})=\beta^{-k}\,k!\,E^k_{(\frac{i\gamma-t}{\beta}+j)}(\frac{i\alpha}{\beta}),\quad i=1,2,3,\quad j=0,1.\tag{5.5}$$

Theorem 5.2 *Let $X:CTGM(\Theta)$. Then, the k th moment of X is given by*

$$E[X^k]=\beta^{-k}\,k!\{(1-\lambda_1)\,e^{\frac{\alpha}{\beta}}\,E^{k-1}_{(\frac{\gamma}{\beta}+1)}(\frac{\alpha}{\beta})+(\lambda_1-\lambda_2)\,e^{\frac{2\alpha}{\beta}}\,E^{k-1}_{(\frac{2\gamma}{\beta}+1)}(\frac{2\alpha}{\beta})$$

$$+\lambda_2\,e^{\frac{2\alpha}{\beta}}\,E^{k-1}_{(\frac{3\gamma}{\beta}+1)}(\frac{3\alpha}{\beta})\}.\tag{5.6}$$

Proof. It is clear that the k th moment is equal to the kth derivative of $M_X(t)$ evaluated at $t=0$ so, to calculate the kth moment of X we replace $t=0$ in (5.4). And using (5.2) for simplifying.

6. CHARACTERISTIC FUNCTION

The characteristic function of the random variable X is (Allan [12]) defined as

$$\phi_X(t)=\mathrm{E}(e^{itX})=\int_{-\infty}^{\infty}e^{itx}\,dF_X(x),\quad i=\sqrt{-1}.\tag{6.1}$$

In this theorem, we will compute the characteristic function of the cubic rank transmuted Gompertz-Makeham probability distribution.

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Theorem 6.1 *Let* $X : CTGM(\Theta)$. *Then the characteristic function of* X, *is*

$$\phi_X(t) = \frac{\gamma(1-\lambda_1)}{\beta} e^{\frac{\alpha}{\beta}} E^0_{(\frac{\gamma-it}{\beta}+1)}(\frac{\alpha}{\beta}) + \frac{2\gamma(\lambda_1-\lambda_2)}{\beta} e^{\frac{2\alpha}{\beta}} E^0_{(\frac{2\gamma-it}{\beta}+1)}(\frac{2\alpha}{\beta})$$

$$+ \frac{3\gamma\lambda_2}{\beta} e^{\frac{3\alpha}{\beta}} E^0_{(\frac{3\gamma-it}{\beta}+1)}(\frac{3\alpha}{\beta}) + \frac{\alpha(1-\lambda_1)}{\beta} e^{\frac{\alpha}{\beta}} E^0_{(\frac{\gamma-it}{\beta})}(\frac{\alpha}{\beta})$$

$$+ \frac{2\alpha(\lambda_1-\lambda_2)}{\beta} e^{\frac{\alpha}{\beta}} E^0_{(\frac{2\gamma-it}{\beta})}(\frac{2\alpha}{\beta}) + \frac{3\alpha\lambda_2}{\beta} e^{\frac{3\alpha}{\beta}} E^0_{(\frac{3\gamma-it}{\beta})}(\frac{3\alpha}{\beta}). \tag{6.2}$$

7. QUANTILE FUNCTION

Theorem 7.1 *Let* $X : CTGM(\Theta)$. *Then the quantile function of* X, *is given by*

$$x_q = \frac{1}{\beta\gamma}[\alpha - \beta \log B(q,\lambda_1,\lambda_2)] - \frac{1}{\beta} p(\frac{\alpha}{\gamma} e^{\frac{\alpha}{\gamma}} B(q,\lambda_1,\lambda_2)^{-\frac{\beta}{\gamma}}),$$

where $p(z)$ is the principle solution for w in $z = we^w$ (or the Lambert W-function).

    Proof. To compute the quantile function of the CTGM distribution, we replace about x by x_q and about $F(x)$ by q in (1.7) to get the equation

$$q = [1 - e^{\frac{\alpha}{\beta}(1-e^{\beta x_q})-\gamma x_q}][1 + \lambda_1 e^{\frac{\alpha}{\beta}(1-e^{\beta x_q})-\gamma x_q} + \lambda_2 e^{\frac{2\alpha}{\beta}(1-e^{\beta x_q})-2\gamma x_q}]. \tag{7.1}$$

    Now we solve the equation (7.1) for x_q. So, let $y = e^{\frac{\alpha}{\beta}(1-e^{\beta x_q})-\gamma x_q}$. Then, (7.1) becomes

$$ay^3 + by^2 + cy + d = 0,$$

where $a = \lambda_2$, $b = (\lambda_1 - \lambda_2)$, $c = (1-\lambda_1)$ and $d = (q-1)$.

    Let the function $B(q,\lambda_1,\lambda_2)$ be defined by

$$B(q,\lambda_1,\lambda_2) = -\frac{b}{3a} - \frac{2^{1/3}\xi_1}{3a(\xi_2+\sqrt{4\xi_1^3+\xi_2^2})^{1/3}} + \frac{(\xi_2+\sqrt{4\xi_1^3+\xi_2^2})^{1/3}}{3(2^{1/3})a},$$

where $\xi_1 = -b^2 + 3ac$, $\xi_2 = -2b^3 + 9abc - 27a^2d$, and $d = 1-q$.

    Hence,

$$e^{\frac{\alpha}{\beta}(1-e^{\beta x_q})-\gamma x_q} = B(q,\lambda_1,\lambda_2). \tag{7.2}$$

Therefor, the solution of (7.2) is

$$x_q = \frac{1}{\beta\gamma}[\alpha - \beta \log B(q,\lambda_1,\lambda_2)] - \frac{1}{\beta} p(\frac{\alpha}{\gamma} e^{\frac{\alpha}{\gamma}} B(q,\lambda_1,\lambda_2)^{-\frac{\beta}{\gamma}}),$$

where $p(z)$ is the principle solution for w in $z = we^w$ (or the Lambert W-function).

8. ORDER STATISTICS

Let $X_1,\cdots,X_n$ be a random sample of size n from the CTGM distribution with parameters $\alpha > 0, \beta > 0, \gamma > 0$, $0 \le \lambda_1 \le 1$ and $\lambda_1 - 1 \le \lambda_2 \le \lambda_1$. Let $X_{1:n}, X_{2:n},\cdots,X_{n:n}$ be the corresponding order statistics obtained by arranging $X_i, i = 1,\cdots,n$, in non-decreasing order of magnitude. The i th element of this sequence, $X_{i:n}$, is called the i th order statistic.

From (Casella and Berger [13], Page 232), the pdf of the i th order statistics is obtain by

$$f_{X_{i:n}}(x) = i\binom{n}{i} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i}. \tag{8.1}$$

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proposition 8.1 *Let* $X_{i:n}$ *be the* i *th order statistic from* $X : CTGM(\Theta)$ *with* $\lambda_1 \neq 0$ *and* $\lambda_2 \neq 0$. *Then, the pdf of the* i *th order statistic is given by*

$$f_{X_{i:n}}(x) = \sum_{j=0}^{n-i}\sum_{s=0}^{i+j-1}\sum_{v=0}^{i+j-1}\sum_{k=0}^{v}(-1)^{j+s} \, i\binom{n}{i}\binom{n-i}{j}\binom{i+j-1}{s}\binom{i+j-1}{v}\binom{v}{k}$$

$$\times \lambda_1^{(v-k)}\lambda_2^k \{\gamma(1-\lambda_1)\, e^{[\frac{(s+v+k+1)\alpha}{\beta}(1-e^{\beta x})-(s+v+k+1)\gamma x]} + 2\gamma(\lambda_1 - \lambda_2)$$

$$\times e^{[\frac{(s+v+k+2)\alpha}{\beta}(1-e^{\beta x})-(s+v+k+2)\gamma x]} + 3\gamma\lambda_2\, e^{[\frac{(s+v+k+3)\alpha}{\beta}(1-e^{\beta x})-(s+v+k+3)\gamma x]}$$

$$+ \alpha(1-\lambda_1)e^{[\frac{(s+v+k+1)\alpha}{\beta}(1-e^{\beta x})-[(s+v+k+1)\gamma-\beta]x]} + 2\alpha(\lambda_1 - \lambda_2)$$

$$\times e^{[\frac{(s+v+k+2)\alpha}{\beta}(1-e^{\beta x})-[(s+v+k+2)\gamma-\beta]x]} + 3\alpha\lambda_2\, e^{[\frac{(s+v+k+3)\alpha}{\beta}(1-e^{\beta x})-[(s+v+k+3)\gamma-\beta]x]} \}. \qquad (8.2)$$

Proof.

$$f_{X_{i:n}}(x) = i\binom{n}{i}f(x)[F(x)]^{i-1}[1-F(x)]^{n-i} \quad \textit{from (8.1)}$$

$$= i\binom{n}{i}f(x)[F(x)]^{i-1}[\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[F(x)]^j], \quad \textit{by binomial theorem}$$

$$= \sum_{j=0}^{n-i}(-1)^j \, i\binom{n}{i}\binom{n-i}{j}f(x)[F(x)]^{i+j-1}, \quad \textit{then use (8.11.7),(8.11.71.8).}$$

Theorem 8.1 *Let* $X_{i:n}$ *be the* i *th order statistic from* $X : CTGM(\Theta)$ *with* $\lambda_1 \neq 0$ *and* $\lambda_2 \neq 0$. *Then the moment-generating of* $X_{i:n}$ *is*

$$M(t) = \sum_{j=0}^{n-i}\sum_{s=0}^{i+j-1}\sum_{v=0}^{i+j-1}\sum_{k=0}^{v}(-1)^{j+s}\, i\binom{n}{i}\binom{n-i}{j}\binom{i+j-1}{s}\binom{i+j-1}{v}\binom{v}{k}$$

$$\times \lambda_1^{(v-k)}\lambda_2^k\{\frac{\gamma(1-\lambda_1)}{\beta}\, e^{\frac{(s+v+k+1)\alpha}{\beta}}\, E^0_{(\frac{[(s+v+k+1)\gamma-t]}{\beta}+1)}(\frac{(s+v+k+1)\alpha}{\beta})$$

$$+\frac{2\gamma(\lambda_1-\lambda_2)}{\beta}\, e^{\frac{(s+v+k+2)\alpha}{\beta}}\, E^0_{(\frac{[(s+v+k+2)\gamma-t]}{\beta}+1)}(\frac{(s+v+k+2)\alpha}{\beta})$$

$$+\frac{3\gamma\lambda_2}{\beta}\, e^{\frac{(s+v+k+3)\alpha}{\beta}}\, E^0_{(\frac{[(s+v+k+3)\gamma-t]}{\beta}+1)}(\frac{(s+v+k+3)\alpha}{\beta})$$

$$+\frac{\gamma(1-\lambda_1)}{\beta}\, e^{\frac{(s+v+k+1)\alpha}{\beta}}\, E^0_{(\frac{[(s+v+k+1)\gamma-t]}{\beta})}(\frac{(s+v+k+1)\alpha}{\beta})$$

$$+\frac{2\gamma(\lambda_1-\lambda_2)}{\beta}\, e^{\frac{(s+v+k+2)\alpha}{\beta}}\, E^0_{(\frac{[(s+v+k+2)\gamma-t]}{\beta})}(\frac{(s+v+k+2)\alpha}{\beta})$$

$$+\frac{3\gamma\lambda_2}{\beta}\, e^{\frac{(s+v+k+3)\alpha}{\beta}}\, E^0_{(\frac{[(s+v+k+3)\gamma-t]}{\beta})}(\frac{(s+v+k+3)\alpha}{\beta})\}, \qquad (8.3)$$

where $-\infty < t < \infty$.

Proof.

$$M(t) = E(e^{tX}) = \int_{-\infty}^{\infty}e^{tx}f_{X_{i:n}}\, dx, \quad \textit{then by (8.2),}$$

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$$= \int_{-\infty}^{\infty} e^{tx} \sum_{j=0}^{n-i} \sum_{s=0}^{i+j-1} \sum_{v=0}^{i+j-1} \sum_{k=0}^{v} (-1)^{j+s} i \binom{n}{i} \binom{n-i}{j} \binom{i+j-1}{s} \binom{i+j-1}{v} \binom{v}{k}$$

$$\times \lambda_1^{(v-k)} \lambda_2^k \{ \gamma(1-\lambda_1) e^{[\frac{(s+v+k+1)\alpha}{\beta}(1-e^{\beta x})-(s+v+k+1)\gamma x]} + 2\gamma(\lambda_1 - \lambda_2)$$

$$\times e^{[\frac{(s+v+k+2)\alpha}{\beta}(1-e^{\beta x})-(s+v+k+2)\gamma x]} + 3\gamma\lambda_2 \, e^{[\frac{(s+v+k+3)\alpha}{\beta}(1-e^{\beta x})-(s+v+k+3)\gamma x]}$$

$$+\alpha(1-\lambda_1) e^{[\frac{(s+v+k+1)\alpha}{\beta}(1-e^{\beta x})-[(s+v+k+1)\gamma-\beta]x]} + 2\alpha(\lambda_1 - \lambda_2)$$

$$\times e^{[\frac{(s+v+k+2)\alpha}{\beta}(1-e^{\beta x})-[(s+v+k+2)\gamma-\beta]x]} + 3\alpha\lambda_2 \, e^{[\frac{(s+v+k+3)\alpha}{\beta}(1-e^{\beta x})-[(s+v+k+3)\gamma-\beta]x]} \}dx.$$

Then we using (5.1) to get the result.

corollary 8.1 *Let* $X_{i:n}$ *be the* i *th order statistic from* $X : CTGM(\Theta)$ *with* $\lambda_1 \neq 0$ *and* $\lambda_2 \neq 0$. *Then the* k *th derivative of the moment-generating of* $X_{i:n}$ *with respect to* t *is*

$$M^{(k)}(t) = \sum_{j=0}^{n-i} \sum_{s=0}^{i+j-1} \sum_{v=0}^{i+j-1} \sum_{u=0}^{v} c(i,j,s,v,u)\beta^{-k} \, k! \, \lambda_1^{(v-u)} \lambda_2^u \times$$

$$\{ \frac{\gamma(1-\lambda_1)}{\beta} e^{\frac{(s+v+u+1)\alpha}{\beta}} E^k_{(\frac{[(s+v+u+1)\gamma-t]}{\beta}+1)} (\frac{(s+v+u+1)\alpha}{\beta})$$

$$+\frac{2\gamma(\lambda_1-\lambda_2)}{\beta} e^{\frac{(s+v+u+2)\alpha}{\beta}} E^k_{(\frac{[(s+v+u+2)\gamma-t]}{\beta}+1)} (\frac{(s+v+u+2)\alpha}{\beta})$$

$$+\frac{3\gamma\lambda_2}{\beta} e^{\frac{(s+v+u+3)\alpha}{\beta}} E^k_{(\frac{[(s+v+u+3)\gamma-t]}{\beta}+1)} (\frac{(s+v+u+3)\alpha}{\beta})$$

$$+\frac{\alpha(1-\lambda_1)}{\beta} e^{\frac{(s+v+u+1)\alpha}{\beta}} E^k_{(\frac{[(s+v+u+1)\gamma-t]}{\beta})} (\frac{(s+v+u+1)\alpha}{\beta})$$

$$+\frac{2\alpha(\lambda_1-\lambda_2)}{\beta} e^{\frac{(s+v+u+2)\alpha}{\beta}} E^k_{(\frac{[(s+v+u+2)\gamma-t]}{\beta})} (\frac{(s+v+u+2)\alpha}{\beta})$$

$$+\frac{3\alpha\lambda_2}{\beta} e^{\frac{(s+v+u+3)\alpha}{\beta}} E^k_{(\frac{[(s+v+u+3)\gamma-t]}{\beta})} (\frac{(s+v+u+3)\alpha}{\beta}) \} \tag{8.4}$$

where $-\infty < t < \infty$ and

$$c(i,j,s,v,u) = (-1)^{j+s} i \binom{n}{i} \binom{n-i}{j} \binom{i+j-1}{s} \binom{i+j-1}{v} \binom{v}{u}.$$

Proof. The proof as proof of Corollary (6.1).

Theorem 8.2 *Let* $X_{i:n}$ *be the* i *th order statistic from* $X : CTGM(\Theta)$ *with* $\lambda_1 \neq 0$ *and* $\lambda_2 \neq 0$. *Then the* k *th moment of* $X_{i:n}$ *is given by*

$$E[X_{i:n}^k] = \sum_{j=0}^{n-i} \sum_{s=0}^{i+j-1} \sum_{v=0}^{i+j-1} \sum_{u=0}^{v} c(i,j,s,v,u)\beta^{-k} \, k! \, \lambda_1^{(v-u)} \lambda_2^u \times$$

$$\{ \frac{(1-\lambda_1)}{(s+v+u+1)} e^{\frac{(s+v+u+1)\alpha}{\beta}} E^{k-1}_{(\frac{(s+v+u+1)\gamma}{\beta}+1)} (\frac{(s+v+u+1)\alpha}{\beta})$$

$$+ \frac{2(\lambda_1 - \lambda_2)}{(s+v+u+2)} e^{\frac{(s+v+u+2)\alpha}{\beta}} E^{k-1}_{(\frac{(s+v+u+2)\gamma}{\beta}+1)}(\frac{(s+v+u+2)\alpha}{\beta})$$

$$+ \frac{3\lambda_2}{(s+v+u+3)} e^{\frac{(s+v+u+3)\alpha}{\beta}} E^{k-1}_{(\frac{(s+v+u+3)\gamma}{\beta}+1)}(\frac{(s+v+u+3)\alpha}{\beta})\}. \tag{8.5}$$

Proof. The proof as proof of Theorem (5.2).

corollary 8.2 *Let* $X_{i:n}$ *be the* i *th order statistic from* $X : CTGM(\Theta)$ *with* $\lambda_1 \neq 0$ *and* $\lambda_2 \neq 0$. *Then the expectation of* $X_{i:n}$ *is*

$$E[X_{i:n}] = \sum_{j=0}^{n-i} \sum_{s=0}^{i+j-1} \sum_{v=0}^{i+j-1} \sum_{u=0}^{v} c(i,j,s,v,u) \frac{1}{\beta} \lambda_1^{(v-u)} \lambda_2^u \times$$

$$\{ \frac{(1-\lambda_1)}{(s+v+u+1)} e^{\frac{(s+v+u+1)\alpha}{\beta}} E^0_{(\frac{(s+v+u+1)\gamma}{\beta}+1)}(\frac{(s+v+u+1)\alpha}{\beta})$$

$$+ \frac{2(\lambda_1 - \lambda_2)}{(s+v+u+2)} e^{\frac{(s+v+u+2)\alpha}{\beta}} E^0_{(\frac{(s+v+u+2)\gamma}{\beta}+1)}(\frac{(s+v+u+2)\alpha}{\beta})$$

$$+ \frac{3\lambda_2}{(s+v+u+3)} e^{\frac{(s+v+u+3)\alpha}{\beta}} E^0_{(\frac{(s+v+u+3)\gamma}{\beta}+1)}(\frac{(s+v+u+3)\alpha}{\beta})\} \tag{8.6}$$

Proof. Substitute about $k = 1$ in (8.5).

9. ENTROPY

9.1. Shannon Entropy

The Shannons entropy ([14]) of a non-negative continuous random variable X with pdf $f(x)$ is defined as

$$H(f) = E[-\log f(X)] = -\int_0^{+\infty} f(x)\log(f(x))dx. \tag{9.1}$$

Below, we are going to use the Expansion of the Logarithm function (Taylor series at 1),

$$\log(x) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{(x-1)^m}{m}, \quad |x| < 1. \tag{9.2}$$

proposition 9.1 *Let* $f(x)$ *be pdf of* $X : CTGM(\Theta)$ *with* $\lambda_1 \neq 1$, $\lambda_2 \neq 0$ *and* $\lambda_1 \neq \lambda_2$. *Then for a real number* θ,

$$f^\theta(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{j} c(i,j,k,\theta)\gamma^{\theta-i} \alpha^i \lambda_2^j (1-\lambda_1)^{\theta-j} (\lambda_1 - \lambda_2)^{j-k} \times$$

$$e^{[\frac{(\theta+3j-k)\alpha}{\beta}(1-e^{\beta x})-[(\theta+3j-k)\gamma-i\beta]x]}, \tag{9.3}$$

where $c(i,j,k,\theta) = \binom{\theta}{i}\binom{\theta}{j}\binom{j}{k} 2^{j-k} 3^j$.

    Proof. From (1.8), we have

$$f^\theta(x) = (\gamma + \alpha e^{\beta x})^\theta e^{\frac{\theta\alpha}{\beta}(1-e^{\beta x})-\theta\gamma x} [1-\lambda_1 + 2(\lambda_1 - \lambda_2)e^{\frac{\alpha}{\beta}(1-e^{\beta x})-\gamma x} + 3\lambda_2 e^{\frac{2\alpha}{\beta}(1-e^{\beta x})-2\gamma x}]^\theta. \tag{9.4}$$

By the binomial series,

$$(\gamma + \alpha e^{\beta x})^\theta = \sum_{i=0}^{\infty} \binom{\theta}{i} \gamma^{\theta-i} \alpha^i e^{i\beta x} \tag{9.5}$$

and

$$(1-\lambda_1 + 2(\lambda_1 - \lambda_2)e^{\frac{\alpha}{\beta}(1-e^{\beta x})-\gamma x} + 3\lambda_2 e^{\frac{2\alpha}{\beta}(1-e^{\beta x})-2\gamma x})^\theta$$

_____

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{\theta}{j}\binom{j}{k} 2^{j-k}\, 3^j\, \lambda_2^j (1-\lambda_1)^{\theta-j}(\lambda_1-\lambda_2)^{j-k}\, e^{[\frac{(3j-k)\alpha}{\beta}(1-e^{\beta x})-(3j-k)\gamma x]}. \tag{9.6}$$

Substitute (9.5) and (9.6) in (9.4) to get the result.

Note: It is clear that when $\theta = n$ natural number in (9.3),

$$f^n(x) = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{j} c(i,j,k,n)\gamma^{n-i}\, \alpha^i\, \lambda_2^j(1-\lambda_1)^{n-j}(\lambda_1-\lambda_2)^{j-k} \times$$

$$e^{[\frac{(n+3j-k)\alpha}{\beta}(1-e^{\beta x})-[(n+3j-k)\gamma-i\beta]x]}, \tag{9.7}$$

where $c(i,j,k,n) = \binom{n}{i}\binom{n}{j}\binom{j}{k} 2^{j-k}\, 3^j$.

Theorem 9.2 *Let $X : CTGM(\Theta)$ with $\lambda_1 \neq 1$, $\lambda_2 \neq 0$ and $\lambda_1 \neq \lambda_2$. Then Shannons entropy of X is given by*

$$H(f) = \frac{1}{\beta}\sum_{m=1}^{\infty}\sum_{n=0}^{m}\sum_{i=0}^{n+1}\sum_{j=0}^{n+1}\sum_{k=0}^{j} c(i,j,k,n,m)\, \gamma^{n+1-i}\, \alpha^i\, \lambda_2^j(1-\lambda_1)^{n+1-j} \times$$

$$(\lambda_1-\lambda_2)^{j-k}\, e^{\frac{\alpha}{\beta}(n+1+3j-k)}\, E^0_{(\frac{(n+1+3j-k)\gamma}{\beta}-i+1)}(\frac{\alpha}{\beta}(n+1+3j-k)), \tag{9.8}$$

where $c(i,j,k,n,m) = (-1)^n \binom{m}{n}\frac{1}{m} c(i,j,k,n+1)$.

Proof. By the Expansion of the Logarithm function (9.2)

$$H(f) = -\int_0^{\infty} f(x)\log(f(x))dx = \int_0^{\infty}\sum_{m=1}^{\infty}\sum_{n=0}^{m}(-1)^n\binom{m}{n}\frac{1}{m} f^{n+1}(x)dx, \tag{9.9}$$

substituting (9.7) in (9.9) to get

$$H(f) = \int_0^{\infty}\sum_{m=1}^{\infty}\sum_{n=0}^{m}(-1)^n\binom{m}{n}\frac{1}{m}\sum_{i=0}^{n+1}\sum_{j=0}^{n+1}\sum_{k=0}^{j} c(i,j,k,n+1)\gamma^{n+1-i}\, \alpha^i\, \lambda_2^j$$

$$\times (1-\lambda_1)^{n+1-j}(\lambda_1-\lambda_2)^{j-k}\, e^{[\frac{(n+1+3j-k)\alpha}{\beta}(1-e^{\beta x})-[(n+1+3j-k)\gamma-i\beta]x]}\, dx,$$

then use (5.1) to get the result.

9.2. Renyi entropy

If X is a non-negative continuous random variable with pdf $f(x)$, then the *Renyi entropy of order θ* (See Renyi [15]) of X is defined as

$$H_\theta(f) = \frac{1}{1-\theta}\log\int_{-\infty}^{+\infty} f(x)^\theta dx, \quad \forall \theta > 0 \ (\theta \neq 1). \tag{9.10}$$

Theorem 9.2 *Let $X : CTGM(\Theta)$ with $\lambda_1 \neq 1$, $\lambda_2 \neq 0$ and $\lambda_1 \neq \lambda_2$. Then the Renyi entropy of order θ of X is given by*

$$H_\theta(f) = \frac{1}{1-\theta}\{-\log\beta + \log[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j} c(i,j,k,\theta)\, \gamma^{\theta-i}\, \alpha^i\, \lambda_2^j(1-\lambda_1)^{\theta-j}$$

$$\times (\lambda_1-\lambda_2)^{j-k}\, e^{\frac{\alpha}{\beta}(\theta+3j-k)}\, E^0_{(\frac{(\theta+3j-k)\gamma}{\beta}-i+1)}(\frac{\alpha}{\beta}(\theta+3j-k))]\}.$$

Proof. To compute $H_\theta(f)$, we substitute (9.3) in (9.10).

9.3. q-Entropy

The q-entropy was introduced by Havrda and Charvat [16]. It is the one parameter generalization of the Shannon entropy. Ullah [17] defined the q-entropy as

$$I_H(q) = \frac{1}{q-1}\{1 - \int_0^\infty f(x)^q \, dx\}, \tag{9.11}$$

where $q > 0$ and $q \neq 1$.

Theorem 9.3 *Let* $X : CTGM(\Theta)$ *with* $\lambda_1 \neq 1$, $\lambda_2 \neq 0$ *and* $\lambda_1 \neq \lambda_2$. *Then q-entropy of* X *is given by*

$$I_H(q) = \frac{1}{q-1}\{1 - \frac{1}{\beta}\sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^j c(i,j,k,q)\gamma^{q-i}\,\alpha^i\,\lambda_2^j(1-\lambda_1)^{q-j}$$

$$\times (\lambda_1 - \lambda_2)^{j-k}\, e^{\frac{\alpha}{\beta}(q+3j-k)} E^0_{(\frac{(q+3j-k)\gamma}{\beta}-i+1)}(\frac{\alpha}{\beta}(q+3j-k))\}.$$

Proof. To find $I_H(q)$, we substitute (9.3) in (9.11).

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