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Real and Imaginary Platform in Mathematics Applied to Science and Technology

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ABSTRACT
In this article we explore the various attractive features of the complex number system as the real and imaginary platform in mathematics. We considered the graphical representation of complex numbers, the properties of functions of complex variables and the paradoxes of the imaginary unit. We wish to discuss the various limit concepts such as complex derivative, complex integrals and their applications in various spheres of human knowledge. We will use transformations in the complex domain to define and use some important functions especially in science and technology.

Key words: complex number, imaginary unit, complex variable, transformation, special function

1. INTRODUCTION

In order to answer the question: "Can we find an operation i such that $i^2 = -1$" complex numbers and operators were invented. We need i as an operation which consist simply in turning $0A = (1, 0)$ through 90^0 or $\pi/2$! When it was realized that the polynomial equation $x^2 + 1 = 0$ has no real number solution and that in fact all negative numbers have no square roots (or no other root at all) in the real number system, it became necessary to introduce and invent the complex number system. The complex numbers and their functions constitute complex variable theory or complex analysis. Complex analysis is one of the most beautiful as well as useful branch of mathematics. Although originating in an atmosphere of mystery suspicion and distrust as evidenced by the terms "imaginary" and "complex" present in the literature it was placed on a sound foundation in the 19^{th} century through the efforts of Cauchy, Riemann, Weierstrass, Gauss and other great mathematicians [9, 10, 11]. Today complex analysis is recognized as an essential part of the mathematical background of engineers, physicists, mathematicians and other scientists. The theory of complex variables is of tremendous value in the solutions of problems of heat flow, potential theory, fluid mechanics, electromagnetic theory, aerodynamics, elasticity and many other fields of science, engineering and technology [7].

2. PRELIMINARIES AND BASIC DEFINITIONS OF TERMS

(1) REAL NUMBER [2]: Any number in the interval $(-\infty, \infty)$ that can be geometrically represented on the real number line.
(2) **COMPLEX VARIABLE**: Any variable in the form $z = x + iy$ where x and y are real variables is a complex variable.
(3) **COMPLEX VALUED FUNCTION**: This is a function whose value is a complex number.
(4) IMAGINARY UNIT: The number defined by $i^2 = -1 \; or \; i = \sqrt{-1}$ is the imaginary unit.
(5) COMPLEX NUMBER: Any number which can be written as $Z = a + ib$ where a and b are real numbers with i as the imaginary unit is a complex number.
(6) ANALYTIC FUNCTION: functions that are differentiable in the neighborhood of a point Z_0 in the complex plane are known as analytic functions.
(7) SINGULARITY: A point in the complex plane in which a point is not analytic is called a singularity.

(8) BASIC AND APPLIED SCIENCES: The basic sciences consist of the subjects such as biology, chemistry, mathematics and physics while the applied sciences include all fields that use the basic sciences. Examples are engineering, technology, medicine, aerodynamics etc.

3. ATTRACTIVE FEATURES IN THE COMPLEX NUMBER SYSTEM [4].

3.1: Representation: Any complex number Z can be represented as a pair of real numbers. This is done by writing $Z = x + iy = (x, y)$ where $i^2 = -1$. In this case x is called the **real part** of Z, y the **imaginary part** of Z and i the **imaginary unit.**

3.2: Geometric diagram: To represent a complex number geometrically we do so on the complex geometrical plane or complex window called Argand diagram. On the Argand diagram the number $Z = x + iy$ is represented as the point $P = (x, y)$ where x is on the real horizontal axis and y is on the imaginary vertical axis. On this diagram the number $i = (0,1)$ can be considered as an operator which turns $(1,0)$ through 90^0 once in the anticlockwise direction.

3.3: Coordinate system representation [5]: Any complex number Z =x+iy can be represented by an ordered pair (x, y) on the Cartesian coordinate system diagram called Argand diagram, after the Swiss mathematician Jean Robert Argand(1768 – 1822). Complex numbers can also be represented by polar coordinates in which case $Z = x + iy = r(cos\theta + i\, sin\theta)$ where $r = \sqrt{x^2 + y^2}$ is the modulus, θ is the angle which $Z(x, y)$ makes with the real axis.The relationship between the complex number $Z = x + iy = r(cos\theta + isin\theta)$ and its n^{th} power is given by the De-Moivres theorem $Z^n = [r(\cos\theta + isin\theta)]^n = r^n(\cos n\theta + isinn\theta)$.

Every complex number $Z = x + iy$ can be represented as a vector OP whose initial point is O(0,0) the origin and terminal point P is the point $P(x, y)$. The dot or scalar product and vector product of such vectors can be appropriately defined.

4. THEORY OF COMPLEX VARIABLES AND FUNCTIONS [6, 7]

Any complex valued function of a complex variable $Z = x + iy$ can be represented by $f(z) = u + iv$ where u is the real part and v is the imaginary part. The definitions and properties of limits, continuity and differentiability of functions hold in the case of functions of complex variables. For functions that are differentiable in the neighborhood of a point Z_0 in the complex plane which are known as **analytic functions** there are some attractive results and properties. The necessary and sufficient condition for such functions to exist is given by the Cauchy Riemann equations which can be stated as follows: A necessary and sufficient condition that $w = f(z) = u(x, y) + iv(x, y)$ is analytic in a region R is that the Cauchy-Riemann equations given by:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial x} \quad \frac{\partial U}{\partial y} = -\frac{\partial v}{\partial x} \qquad (1)$$

are satisfied in R where these partial derivatives are contInuous in R. If we consider a complex function z(t) as a position vector we can define its vector functions gradient, divergence and curl. We can also show that both the real and imaginary parts of an analytic function φ are harmonic or satisfies the Laplace equation given by:

$$\frac{\partial 2\varphi}{\partial x 2} + \frac{\partial 2\varphi}{\partial y 2} = 0 \qquad (2)$$

With the above properties we can apply complex functions to mechanics and geometry.

4.1: Integrals and Series OF Complex Functions

We can evaluate and analyze integrals, sequences and series of functions of complex variables using various results, formulas and procedures in complex analysis. To evaluate series of analytic and non analytic functions we use Taylor and Laurent's series expansions respectively. To evaluate integrals of functions of complex variables we use Cauchy's theorem, Cauchy's integral formula, residue theorem and related substitution theorems. The theorems are given as follows [4, 7]

4.1.1: Cauchy's Theorem

Let f(z) be analytic in a region R and on its boundary C. Then

$$\oint_c f(z)dz = 0 \qquad (3)$$

4.1.2: Cauchy's Integral Formula

If f(z) is analytic inside and on the boundary C of a simply-connected region *R*, then Cauchy's integral formula states that:

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz \qquad (4)$$

4.1.3: Laurent's Theorem

Let f(z) be single-valued and analytic inside and on a circle C except at the point z=a chosen as the centre of C. Then f(z) has a Laurent series about z=a given by:

$$(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \ldots \qquad (5)$$

$$a_n = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{n+1}}dz \qquad n = 0, \pm 1, \pm 2, \ldots \qquad (6)$$

If n = -1 we have $\oint_c f(z)dz = 2\pi i a_{-1}\ldots$ (7)

Generally we get that:

The integral $\oint_c \frac{dz}{(z-a)^p} = \begin{cases} 2\pi i & p = 1 \\ 0 & p = integer \neq 1 \end{cases} \ldots$ (8)

The coefficient a_{-1} is called the residue of f(z) at z = a. The concept of residues is very important in the evaluation of integrals as indicated in the residue theorem which we state as follows:

4.1.4: The Residue Theorem

Let f(z) be single-valued and analytic inside and on a simple closed curve C except at the singularities a, b, c, ……inside C which have residues given by a_{-1}, b_{-1}, c_{-1}, ……Then the residue theorem states that:

$\int_C f(z)dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \ldots\ldots\ldots). \ldots$ (9)

Various integrals useful in mathematics and science can be evaluated using the residue theorem [7].

5. CONFORMAL MAPPING, SPECIAL FUNCTIONS AND APPLICATIONS

The set of equations $u = u(x,y), v = v(x,y)$ defines a transformation or mapping which establishes a correspondence between points in the u-v and x-y planes. A transformation which preserves angles in both magnitude and sense is called a conformal transformation. The mathematics of conformal mappings have found applications in physics and engineering especially in the solution of boundary value problems in fluid dynamics, electrostatics and heat flow. In finding solutions to equations in mathematical physics we use special integral functions of complex variables such as Bessel functions, Neumann's functions, hyper-geometric function, gamma function, error function and the elliptic function. Complex valued functions are used in science and engineering to describe vibratory motion, harmonic oscillations, damped vibrations, alternating currents and various wave phenomena especially in quantum and elementary particle physics [7, 11]. Thus the usefulness of complex numbers and variables can never be over emphasized. Functions of complex variables are today used in the analysis of electronic circuitry of wing lift in aerodynamics as well as water seepage dams [1].

5.1: **Complex Variables in Classical and Quantum Mechanics**

In classical mechanics the concepts of forces, positions, moments, potentials, electric and magnetic fields are all real quantities and expressible as complex numbers with zero imaginary part. The equations describing the above quantities such as Newton's laws, Maxwell's equations etc are all differential equations involving purely real quantities. In quantum mechanics quantities and functions contains the imaginary unit $i = \sqrt{-1}$, for example the Schrodinger's equation of motion for atomic and sub-atomic particles is given by:

H$\varphi = E\varphi$ where H is Hamiltonian, E is eigen energy and φ is wave function. The famous form of the quantum mechanical Hamiltonian operator is given by:

H = $\frac{\dot{p}^2}{2m}$ + V($\dot{x}$) = ih$\frac{\partial}{\partial t}$ where i is the imaginary unit and h is Plank's constant [8].

6. PARADOXES OF THE IMAGINARY UNIT

6.1: The position of i with respect to zero [3]

We know that $i \neq 0$ since by definition $i^2 = -1$. So $i > 0$ or $i < 0$ which in both case implies $i^2 > 0$ which we know is not true, i.e the imaginary unit i is neither less than zero or greater than zero.

Paradox: Is it possible for a non-zero number to be neither greater than zero nor less than zero?

6.2: The imaginary unit i is not real

The imaginary unit i is not a real number but an imaginary number whose square is a real number, that is $i^2 = -1$.

Paradox: How can you multiply something that is not a real number by itself to get a real number? Are we not supposed to reap what we sow?

6.3: The multiplicative power of i is an algebraic group

The powers of i can be given beautifully as follows:

$i^n =$ i for n = 1, 5, 9, 13……………..(All odd powers)

$= -1$ *for* n = 2, 6, 10, 14……….(All even powers)

$= -i$ for n = 3, 7, 11, 15………..(All odd powers)

$= 1$ for n = 4, 8, 12, 16……………(All even powers)

The set $\{i, -i, 1, -1\}$ is an algebraic multiplicative group. No number except 1 has the set of all its powers as a multiplicative group and yet $i \neq 1$.

Paradox: Is i really a number?

7. CONCLUSION

We have explored the complex number system with its treasures and attractive properties. We have seen that the complex numbers and functions are useful in the solutions of problems in mathematics, physical sciences, engineering and technology. We also presented some paradoxical remarks on the imaginary unit which is the fundamental building block of the complex number system.

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