Available online www.ejaet.com

European Journal of Advances in Engineering and Technology, 2018, 5(10): 806-809

Research Article ISSN: 2394 - 658X

Fractal Geometry Applied to Biological and Botanical Systems

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ABSTRACT

Fractal geometry is the geometry of most objects occurring in nature and society. To describe complicated and unconventional geometrical objects like a tree or the coast of a river or the structure of human organs like the lungs and the brain we use fractal geometry. In this article we use fractal geometry to describe biological systems such as the heart, the kidney and the veins as well as botanical systems such as the structure of plants and flowers. The botanical systems can be analyzed using the golden concepts associated with the Fibonacci sequence which are the golden mean and the golden angle. This paper will discuss fractal geometry applied to vital human organs as well as the applications of the golden mean and angle in the botanical systems especially in the fractal nature and structure of plants and flowers.

Key words: Biological system, Botanical system, Fractal geometry, Fibonacci sequence

1. INTRODUCTION

Fractal geometry is the appropriate geometry that can be used to describe complex, complicated and unconventional shapes and structures in both natural and artificial systems. To understand life, objects, structures and behavior of systems around us we must appreciate chaos, fractals and turbulence. To explain, analyze and understand fractal and chaotic behavior in both natural and artificial systems we use the mathematics of chaos theory. Chaos theory is a scientific discipline which is focused on the study of nonlinear systems, which are generally complex and unpredictable [3]. The cause of unpredictability in nonlinear systems is extreme sensitivity to initial conditions-what is referred to as the butterfly effect. The concept means that with a complex non-linear system, very small changes in the starting conditions of a system will result in dramatically different and large changes in the outputs for that system. Chaos is an unpredictable behavior occurring in a deterministic system [1]. Chaos which is caused by extreme sensitivity to initial conditions in the parameters of a system explains the complexity, the dual unpredictability and determinism in a lot of physical and social systems. The property of sensitivity to initial conditions can be quantified as:

$$|\delta x(t)| \approx e^{\lambda t}|\delta x(0)| \quad (1)$$

Where λ, *the mean* rate of separation of trajectories of the system in the fractal diagram [9]. In this article we will illustrate the relationship between fractal geometry and natural systems by considering the fractal properties of the golden mean and the golden angle associated with the Fibonacci sequence. Fractal geometry which is referred to as the geometry of nature can be used to adequately describe the fractal properties, structure and nature of plants and flowers.

2. DEFINITION AND EXPLANATION OF RELEVANT TERMS

(1) Chaos is the behavior of a dynamical that has large number of attractors and is sensitive to initial conditions.
(2) Fractal is an irregular shape with self-similarity and fractional dimension.
(3) Fractal dimension is measure (or concept of length, area or volume) of a geometric object that can take on fractional non-integer values.

(4) Self-similarity is the property of objects which looks the same on magnification or have the property of repetition of structures at different length scales. For example fractal objects like the Cantor set and the Sierpenski's triangle have such property.

(5)The Fibonacci recursive formula or difference equation is given by $F_{n+1} = F_n + F_{n-1}$, $F_1 = F_2 = 1$ for ($n \geq 2$).The Fibonacci difference equation defines a famous sequence whose terms are called the Fibonacci numbers. The first few terms are given are given by:

1, 1, 2, 3,5 ,8, 13, 21, 34.

(6) Sensitivity to initial conditions can be quantified as:

$|\delta x(t)| \approx e^{\lambda t}|\delta x(0)$

Where λ, *the mean* rate of separation of trajectories of the system, is called the lyapunov exponent.

3. PROPERTIES OF OBJECTS WITH FRACTAL FEATURES

A fractal is a complicated geometric figure that, unlike a conventional complicated figure, does not simplify when it is magnified. Fractal geometry is used to describe trajectories and structures occurring in nature, science and society especially those produced by chaotic dynamical systems. The term "fractal" was coined in the 1960's by B. Mandelbrot, a mathematician at IBM. The word 'fractal' is derived from the Latin word *fractus,* meaning broken or fragmented. It is generally acknowledged that fractals have some or all of the following properties: complicated structure at a wide range of length scales, repetition of structures at different length scales (self-similarity), and a fractal dimension that is not an integer. Among the geometrically constructed mathematical fractal objects are the Cantor set, the Sierpinski triangle, Koch curve and the fractal cube. Mathematically a fractal is a set with a fractional and non-integral dimension greater than its dimension. A fractal can also be described as a function which is continuous, yet non-differentiable and is auto-correlated over a range of scales. Fractals possess the property of self-similarity. Their structure (e.g statistical roughness) looks similar at all scales of observation. Fractals are typically parameterized by their fractal dimension D which is always greater than the Euclidean dimension. Fractals occupy more space than their Euclidean counterparts. The following table gives the fractal dimensions of some basic mathematical fractal structures:

Table -1 Fractal dimensions of some fractals [2]

Cantor set	ln2/ln3
Koch carpet	ln4/ln3
Sierpinski carpet	ln8/ln3
Fractal cube	ln6/ln2
Menger sponge	ln20/ln3

4. FRACTAL GEOMETRY APPLIED TO BIOLOGICAL SYSTEMS

Many biological systems present fractal aspects of their structure and dynamics from molecular level to the entire organism. At molecular level, it has been demonstrated that proteins backbones and surfaces are fractals. At cell level, there are also many evidences of fractal organization of different cell types which is also true for both plant and animal tissues. At the organism level there are studies providing the role of non-linear phenomena in sustaining life. In the human body vital organs, systems and processes can be best and appropriately described by fractal geometry. For example the geometric shape and architecture of the blood vessels network, the lung and heart as well as tissues and membranes shows that there are fractals in nature and structure. The following table gives the fractal dimensions for some cellular media.

Table -2 Fractal Dimension Of Some Cellular Media

Serial Number	Cellular Media	Fractal Dimension
1	Alveolar membrane	2.17
2	Mitochondria membrane	2.09 /2.52
3	Mitochondria membrane contour	1.54
4	Plasma membrane contour	1.07+1.43
5	Nucleus membrane contour	1.027
6	Plant nucleus (tomato and green pepper)	1.5/1.8
7	Neuron cytoplasm	1.47/1.7
8	Reticulom endoplasm	1.72

The human body is a complex dynamic system that plays by the rules of chaos and fractal geometry. The distributive systems of the body which include cardiovascular, respiratory, lymphatic, digestive and excretory all display fractal characteristics. For example, the branching of arteries, veins and capillaries in the cardiovascular system is random, self – similar and detailed at every scale. From the strange attractors in a heartbeat to the fractal dimensions in the lungs we observe fractal features [2].

The following is a biological model of population fluctuation in the flour beetle *Tribolium.* The newly hatched larva spends two weeks feeding before entering a pupa stage of about the same length. The beetle exits the pupa stage as an adult. Let the numbers of larvae, pupae, and adults at any given time t be denoted by L_t, P_t, and A_t respectively. After the unit time of two weeks the model for the three beetle populations is given by:

$L_{t+1} = bA_t$ (2)

$P_{t+1} = L_t(1- \mu_l)$ (3)

$A_{t+1} = P_t(1- \mu_p) + (1- \mu_a)$ (4)

Where b is the birth rate of the species (the number of new larvae per adult after each unit time), μ_l, μ_p *and* μ_a are the death rates of the larvae, pupa, and adult, respectively. The above is a three dimensional discrete map. *Tribolium* adds an interesting twist to the above model: cannibalism caused by overpopulation stress. Under conditions of overcrowding, adults, adults will eat pupae and unhatched eggs (future larvae); larvae will also eat eggs. Incorporating these into above model yields:

$L_{t+1} = bA_t\exp(- c_{ea}A_t – c_{el}L_t)$ (5)

$P_{t+1} = L_t(1- \mu_l)$ (6)

$A_{t+1} = P_t(1- \mu_p)\exp(-c_{pa}A_t) + A_t(1- \mu_a)$ (7)

From population experiments we get the following parameter values:
$c_{el} = 0.012$, $c_{ea} = 0.009$, $c_{pa} = 0.004$, $\mu_l = 0.267$, $\mu_p = 0$ and b = 7.48 with mortality rate of the adult as μ_a = 0.0036. From a graph of adult mortality rate against larvae population we get that for relatively low mortality rates, the larvae population reaches a steady or equilibrium state (fixed point). For $\mu_a > .1$ (representing a death rate of 10% of the adults over each two week period), the model shows oscillation between two widely different states. This is a 'boom' and 'bust' cycle [1]. A low population (bust) leads to uncrowded living conditions and increased growth (boom) at the next generation which eventually lead to overcrowding, cannibalism, catastrophic decline and then a repeat of the cycle. A bifurcation is a change from a fixed population to an oscillation between high and low populations. For the above model the population changes from period-doubling birfurcation (near $\mu_a = 0.1$) to period-halving birfurcation (when $\mu_a \approx 0.6$) and then to chaos (near $\mu_a = 1$) or adult death rate of 100%. The bifurcation diagram is a fractal structure.

5. FRACTAL GEOMETRY APPLIED TO BOTANICAL SYSTEMS

5.1. The Fibonacci Sequence

One way to define a sequence is to give a recursive formula or difference equation. The Fibonacci recursive formula or difference equation is given by:

$F_{n+1} = F_n + F_{n-1}$, $F_1 = F_2 = 1$ for ($n \geq 2$) (8)

The difference equation (8) defines a famous sequence whose terms are called the Fibonacci numbers. The first few terms are given are given by:

1, 1, 2, 3,5 ,8, 13, 21, 34. We will solve (8) by the characteristic equation method as follows:

$F^2 – F – 1 = 0$ is the associated characteristic equation which has the roots given by: $F_1 = \frac{1+\sqrt{5}}{2}$ or $F_2 = \frac{1-\sqrt{5}}{2}$.

The general solution is given by $F_n = K_1(\frac{1-\sqrt{5}}{2})^n + K_2(\frac{1+\sqrt{5}}{2})^n$ (9)

Where $K_1 = F_1/\sqrt{5}$ and $K_2 = - F_2/\sqrt{5}$ which gives the required solution as:

$F_n = \frac{1}{\sqrt{5}}\{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1}\}$ for n = 0, 1, 2,.. (10)

The Fibonacci sequence was encountered by Fibonacci also known as Leonardo of Pisa (circa 1175-1250) as a problem involving the number of offspring of rabbits. The Fibonacci sequence models the number of rabbits expected at the end of every month starting with a pair of rabbits which can produce a pair every month which in turn becomes productive after every month**.** In particular the Fibonacci mathematical series was discovered in 1209 by Leonardo of Pisa, known as Fibonacci while computing the total number of adult rabbits in successive months starting with a single adult rabbit pair and assuming that each adult rabbit pair produces one pair of offspring each month and that baby rabbit pairs became adults in one month time [9].

5.2: Golden Concepts in Fibonacci sequence and Fractal Geometry of Plants and Flowers

The ratio of adjacent elements of the Fibonacci sequence approaches the irrational number $\tau = (1 + \sqrt{5})/2$ $\cong 1.618$ in the limit. The number $\tau, is\ the$ solution to the algebraic equation $1+x = x^2$ which implies that $1+\tau = \tau^2$. Thus the double geometric sequence given by:

$\frac{1}{\tau^3}, \frac{1}{\tau^2}, \frac{1}{\tau}, 1, \tau, \tau^2, \tau^3, ...$ (11)

is the Fibonacci sequence since it has the property that each term is equal to the sum of the earlier two terms and also the ratio of each term to the earlier term is equal to the **golden mean** $\tau. It$ is the only geometric series which is also a Fibonacci sequence. Self-similar structures have in their geometrical design the noble numbers (numbers which are functions of the golden mean) which are all characterized by five-fold symmetry of the pentagon and dodecahedron. For example, the ratio of the length of the diagonal to the side in a regular pentagon is equal to the golden mean. The commonly found shapes in nature are the helix and the dodecahedron which have signatures of self-similarity underlying Fibonacci numbers. The branching network in the summation process of the Fibonacci sequence is a hierarchy of self-similar networks which are fractals. The association of the **golden mean** and the **golden angle** with growth of self-similar patterns has been established quantitatively in plant phyllotaxis in botany. Phyllotaxis is the study of the arrangement of all plant elements, which originate as primodia on the short apex. The botanical elements of a plant include branches, roots, leaves, petals, stamens, sepals, florets and flowers. These plant elements begin their existence as primodia in the neighborhood of the undifferentiated shoot apex (extremity). Observations in botany show that in about 92% of plants studied worldwide, premodia emerge as protuberances at locations such that the angle subtended at the apical centre by two successive primodia is equal to the **golden angle** $\varphi = 2\pi(1 - \frac{1}{\tau})$ corresponding to approximately 137.5 degrees [2].

6. SUMMARY AND CONCLUSION

Fractals describe- non Euclidean objects generic to nature such as tree roots, tree branches, river basins and human vital organs such as the heart, the lungs, the brain and the kidneys. The tree structure generated by the process of getting the Fibonacci sequence has been shown to be a fractal with golden concepts of golden mean and golden angle. The golden concepts were shown to be significant in science and nature, especially in the fractal geometry of botanical structures such as plants and flowers as well as mathematical polygonal figures. Fractal structure, nature and behavior can be observed in many biological and botanical systems and shall always be a mystery, a paradox, a puzzle, an enigma and a riddle in natural and scientific research.

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