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Solution of Second Order Initial- Boundary Value Problems of Partial Integro-Differential Equations by using a New Transform: Mahgoub Transform

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ABSTRACT

The partial integro-differential equations (PIDEs) have many possible applications in areas like mathematics, physics and engineering. Therefore, we develop a new transform, which was proposed by Mahgoub [1], for solving second order initial-boundary value problems (IBVPs) of PIDEs. This transform is characterized by its simplicity of use.

Key words: Integral transform, Mahgoub transform, Boundary value problems, Partial integro-differential equations.

INTRODUCTION

Let's look at the second order IBVP of PIDE as follows:

$$u_{tt}(x,t)=u_x(x,t)+\lambda\int_0^t k(t-s)u(x,s)ds+g(x,t),\quad x\in\Omega,\ t\in J \tag{1}$$

with initial (IC) and boundary (BC) conditions

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_{t_0}(x),\qquad x\in[a,b]$$
$$u(a,t)=h(t),\qquad u(b,t)=l(t),\qquad t\in[0,T],$$

where $u_{tt}(x,t)=\frac{\partial^2 u}{\partial t^2}, u_x(x,t)=\frac{\partial u}{\partial x}, \lambda$ is constant, $g(x,t)$ is known function and $k(t-s)$ is given kernel function.

Several methods for solving PIDEs are given [2-9].

In this paper, we present application Mahgoub transform for solving second order IBVPs of PIDEs.

APPLICATION MAHGOUB TRANSFORM TO SECOND ORDER IBVPS OF PIDEs

Recently, a new transform was proposed by Mahgoub [1] in 2016. He define the function A for $t\geq 0$ as

$$A=\left\{ f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}} \right\}, \tag{2}$$

where M , k_1,k_2 are constants and M is a finite.

The operator $M\{f(t)\}$ is given as

$$M\left\{f\left(t\right)\right\} = H(v) = v\int_{0}^{\infty} f(t)e^{-vt}dt,\ k_1 \le v \le k_2. \qquad (3)$$

FACT: (LINEARITY PROPERTY)

Let $f(t)$ and $g(t)$ are functions whose Mahgoub transform exists, then

$M\left\{c_1 f\left(t\right)+c_2 g\left(t\right)\right\} = c_1 M\left\{f\left(t\right)\right\}+c_2 M\left\{g\left(t\right)\right\}$, where c_1 and c_2 are constants.

Proof: $M\left\{c_1 f\left(t\right)+c_2 g\left(t\right)\right\} = v\int_{0}^{\infty}\left[c_1 f(t)+c_2 g(t)\right]e^{-vt}dt$

$$= vc_1\int_{0}^{\infty} f(t)e^{-vt}dt + vc_2\int_{0}^{\infty} g(t)e^{-vt}dt$$

$$= c_1 M\left\{f\left(t\right)\right\}+c_2 M\left\{g\left(t\right)\right\}. \ \blacksquare$$

Theorem: Let $f(t)$ and $g(t)$ are functions and given by $M\left\{f(t)\right\} = F\left(v\right)$ and $M\left\{g\left(t\right)\right\} = G\left(v\right)$ then

$$M\left\{f\left(t\right)g\left(t\right)\right\} = \frac{1}{2}F\left(v\right)G\left(v\right).$$

Proof: We choose $F\left(v\right) = v\int_{0}^{\infty} f(w)e^{-vw}dw$ and $G\left(v\right) = v\int_{0}^{\infty} g(l)e^{-vl}dl$, then

$$F\left(v\right)G\left(v\right) = v^2\int_{0}^{\infty}\int_{0}^{\infty} f(w)g(l)e^{-v(w+l)}dldw$$

Let $w+l = t \Rightarrow dw = dt$, then

$$F\left(v\right)G\left(v\right) = v^2\int_{0}^{\infty}\int_{0}^{t} f(w)g(t-w)e^{-vt}dldt$$

$$= vM\left\{f\left(t\right)\ g\left(t\right)\right\}$$

So

$$\frac{1}{v}F\left(v\right)G\left(v\right) = M\left\{f\left(t\right)\ g\left(t\right)\right\}. \ \blacksquare$$

Now, we going to solve (1) by taking $M\{.\}$ on both sides

$$M\left\{u_{tt}\left(x,t\right)\right\} = M\left\{u_x\left(x,t\right)+\lambda\int_{0}^{t} k(t-s)u(x,s)ds+g(x,t)\right\}.$$

By using linearity property, we have

$$M\left\{u_{tt}\left(x,t\right)\right\} = M\left\{u_x\left(x,t\right)\right\}+\lambda M\left\{\int_{0}^{t} k(t-s)u(x,s)ds\right\}+M\left\{g\left(x,t\right)\right\}.$$

From Mahgoub transform formula, it holds that

$$v\int_{0}^{\infty} e^{-vt}\frac{\partial^2 u}{\partial t^2}(x,t)dt = v\int_{0}^{\infty} e^{-vt}\frac{\partial u}{\partial x}(x,t)dt+\lambda\frac{1}{v}\left(\bar{k}\left(v\right)H\left(x,v\right)\right)+\bar{g}\left(x,v\right).$$

$$\Rightarrow v^2 H\left(x,v\right)-v^2 u\left(x,0\right)-vu_t\left(x,0\right) = \frac{d}{dx}H\left(x,v\right)+\lambda\frac{1}{v}\left(\bar{k}\left(v\right)H\left(x,v\right)\right)+\bar{g}\left(x,v\right),$$

where $H\left(x,v\right) = M\left\{u\left(x,t\right)\right\}$, $\bar{k}\left(v\right) = M\left\{k\left(t-s\right)\right\}$, and $\bar{g}\left(x,v\right) = M\left\{g\left(x,t\right)\right\}$.

By substituting IC into above equation, then the solution becomes

$$v^2 H\left(x,v\right)-v^2 u_0-vu_{t_0}\left(x,0\right) = \frac{d}{dx}H\left(x,v\right)+\lambda\frac{1}{v}\left(\bar{k}\left(v\right)H\left(x,v\right)\right)+\bar{g}\left(x,v\right).$$

$$\Rightarrow \frac{d}{dx} H(x,v) + \lambda \frac{1}{v}\left(\bar{k}(v) H(x,v)\right) - v^2 H(x,v) = \bar{g}(x,v) - v^2 u_0 + v u_{t_0}(x,0).$$

This will give linear first order ODE as

$$\frac{d}{dx} H(x,v) + \left(\frac{\lambda}{v^2} - v^2\right) H(x,v) = \bar{g}(x,v) - v^2 u_0 + v u_{t_0}(x,0).$$

We can easily solve this type of ODEs and then find inverse Mahgoub transform.

EXAMPLE

Consider the IBVP [5]

$$u_{tt} = u_x + 2\int_0^t (t-s) u(x,s) ds - 2e^x,$$

(4)

with IC: $u(x,0) = e^x,\ u_t(x,0) = 0.$

BC: $u(0,t) = \cos t.$

To solve (4), take Mahgoub transform on both sides

$$M\{u_{tt}\} = M\left\{u_x + 2\int_0^t (t-s) u(x,s) ds - 2e^x\right\}. \tag{5}$$

By using linearity property, we get

$$M\{u_{tt}\} = M\{u_x\} + 2M\left\{\int_0^t (t-s) u(x,s) ds\right\} - 2e^x M\{1\}. \tag{6}$$

From Mahgoub transform formula, we have

$$v^2 H(x,v) - v^2 u(x,0) - v u_t(x,0) = H_x(x,v) + \frac{2}{v^2}\left(H(x,v)\right) - 2e^x. \tag{7}$$

By substituting IC into (7), we obtain

$$v^2 H(x,v) - v^2 e^x = H_x(x,v) + \frac{2}{v^2}\left(H(x,v)\right) - 2e^x.$$

$$\Rightarrow H_x(x,v) - v^2 H(x,v) + \frac{2}{v^2}\left(H(x,v)\right) + v^2 e^x - 2e^x = 0.$$

$$\Rightarrow H_x(x,v) + \left(\frac{2}{v^2} - v^2\right) H(x,v) = \left(2 - v^2\right) e^x. \tag{8}$$

This will give linear first order ODE as

$$\frac{dH(x,v)}{dx} + \left(\frac{2}{v^2} - v^2\right) H(x,v) = \left(2 - v^2\right) e^x. \tag{9}$$

We can use integration factor $\mu = e^{\int\left(\frac{2}{v^2} - v^2\right)dx}$ to solve (9).

Therefore

$$\bar{H}(x,v) = \frac{\int \mu(x)\left(\left(2 - v^2\right) e^x\right) dx + c}{\mu(x)}. \tag{10}$$

The solution of (10) is

$$\bar{H}(x,v) = \frac{v^2}{1+v^2} e^x + c e^{-\left(\frac{2}{v^2} - v^2\right)x}. \tag{11}$$

Substitute $u(x,0) = e^{x}$ and $u_t(x,0) = 0$ into (11), then $c = 0$.

So, we now have

$$\overline{H}\left(x,v\right) = \frac{v^2}{1+v^2}\,e^{x}. \qquad (12)$$

From the inverse Mahgoub transform, the solution is then

$$H\left(x,v\right) = e^{x}\cos t. \qquad (13)$$

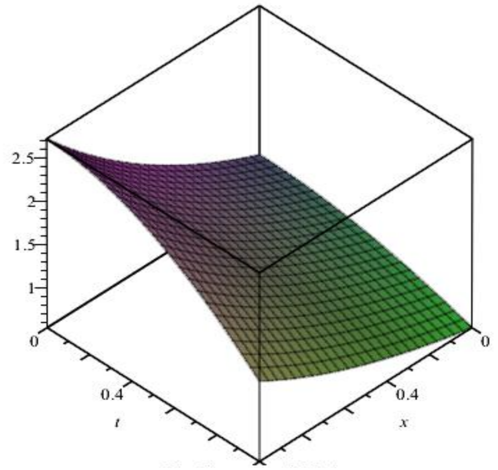

Fig. 1 The graph of H(x,t)

CONCLUSION

Mahgoub transform is characterized by its simplicity of use. Also, it is accurate and efficient technique for finding solution IBVPs of PIDEs.

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